1. Show that in any CNF grammar all parse trees for strings of length n have 2n-1 interior nodes (note that interior nodes have labels which are variables).

   Solution: Consider a parse tree for a string w of length n. If we omit all productions $A \to a$, then we are left with a full binary tree, consisting of n leaves (corresponding to the n terminal symbols of w). It is well-known that a binary tree with n leaves has $n-1$ internal nodes. Thus we have a total of $2n-1$ variables, if we restore the original parse tree.

2. Use the CYK algorithm to check if the strings baaab, aabab are in the language of the grammar

$$
\begin{gathered}
S \to AB|BC \\
A \to BA|a \\
B \to CC|b \\
C \to AB|a
\end{gathered}
$$

   Solution: Straightforward

3. Is there an algorithm for the following decision problem: Given a context-free grammar G, is L(G) finite ?

   Solution: The solution to this is identical to the same decision problem for regular languages.

4. Is there a decision algorithm for the following problem: Given a context-free grammar G, does L(G) contain at least 100 strings ? (Hint: Use the solution to the previous problem)

   Solution: We use the decision algorithm for the previous problem to dtermine if $L(G)$ is finite or infinite.

   If it is infinite then $L(G)$ has at least 100 strings. Otherwise, all strings are of length $\leq n$, where n is the pumping lemma constant. We now generate strings of length 0, 1, 2, etc. up to n, and check for membership, using the CYK algorithm.

5. Show that the following two languages are context-free by giving grammars for each.

$$L_1 = \{a^n b^{2n} c^m | n, m \geq 0\} \; L_2 = \{a^n b^m c^{2m} | n, m \geq 0\}$$

   Is $L_1 \cap L_2$ context-free? Justify your answer.

   Solution:

   The following grammar generates L_1:

$$
\begin{gathered}
S \to XY \\
X \to AXBB|\epsilon \\
Y \to cY|\epsilon \\
A \to a \\
B \to b
\end{gathered}
$$

Similarly, we can construct a grammar to generate L_2.

$L_1 \cap L_2 = \{a^n b^{2n} c^{4n}\}$. This is not context-free, as can be seen by a strightforward use of the Pumping Lemma.

6. Use the pumping lemma to show that the language $L = a^j b^j c^i | j \neq i$ is not context-free.

   Solution: We have to use Ogden's lemma for this one

7. Give context-free grammars that generate the following languages: (a) $L = \{0^i 1^j | i \geq j \geq 0\}$ (b) $L = \{0^i 1^j | i \leq j\}$.

   Solution: The following grammar generates the language in (a)

$$
\begin{gathered}
S \to XY \\
Y \to 0Y1|\epsilon \\
X \to 0X|\epsilon
\end{gathered}
$$

   Similarly for the language in (b)