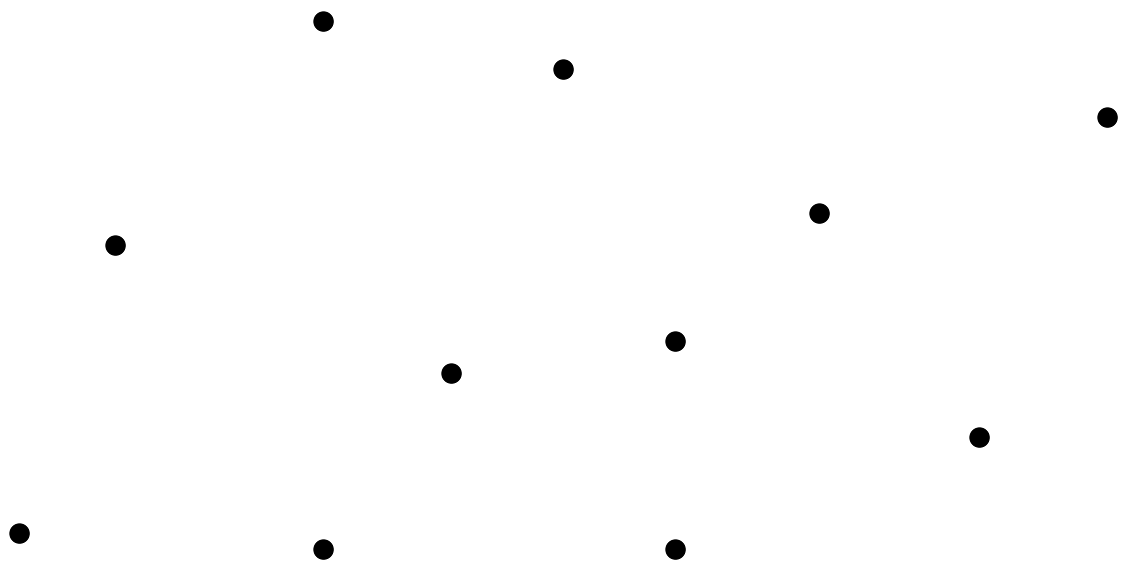
All-maximum and all-minimum problems under some measures

Asish Mukhopadhyay & Satish Panigrahi
University of Windsor, Canada

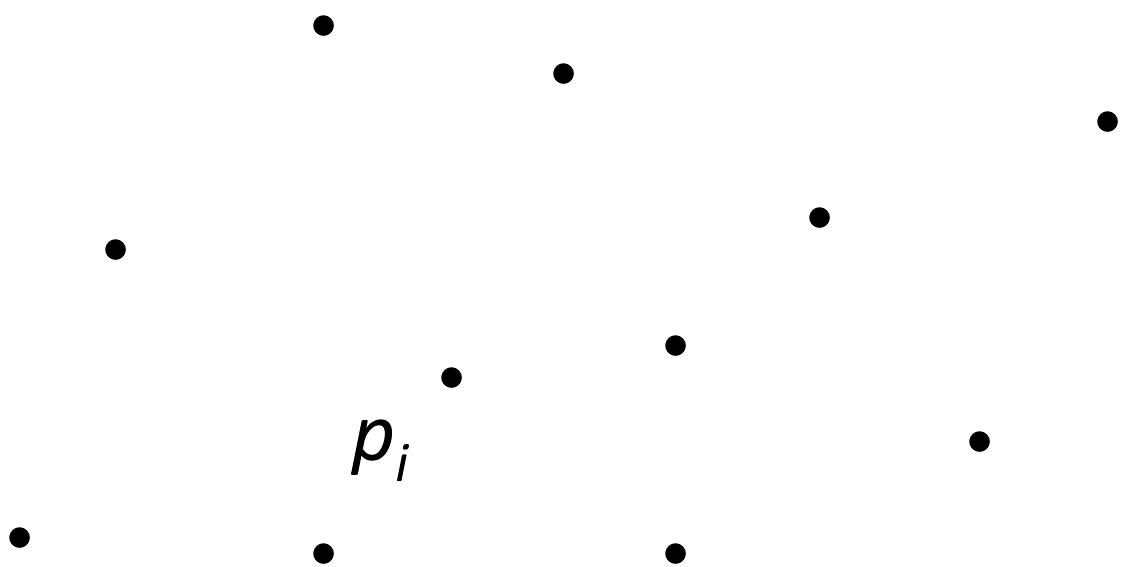
Objective

- Given:



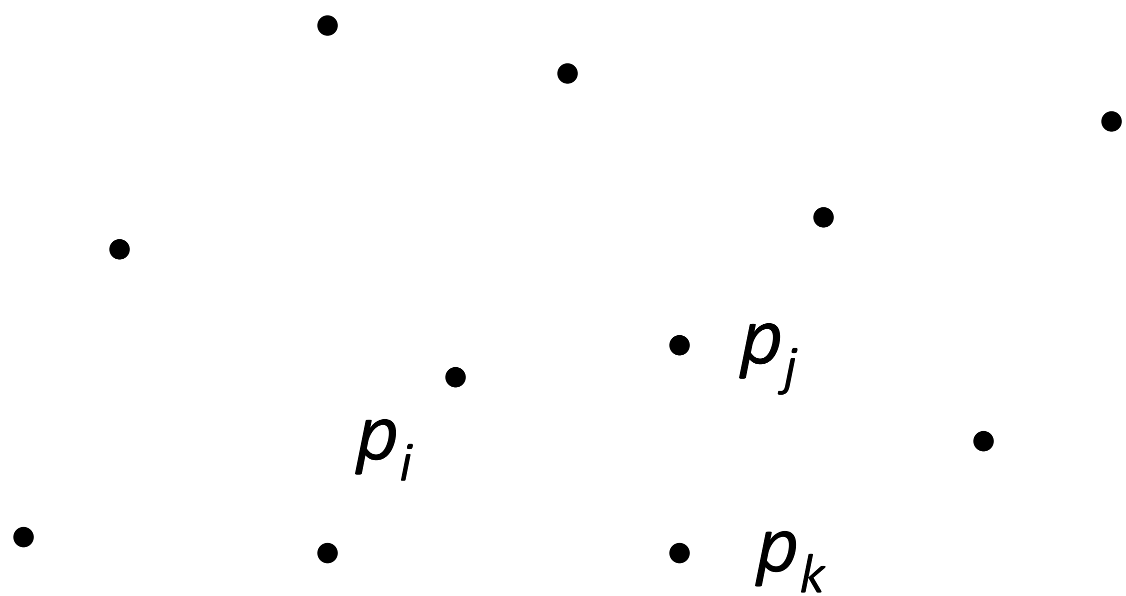
Objective

- For each



Objective

- To *min* or *max* a measure $\mathcal{M}$ on p_i, p_j, p_k



Motivation

- Open problems posed in Duffy et al. [2005], Mukhopadhyay et al [2006] , Daescu et al. [2006]
- 2-point site Voronoi diagrams studied by Barequet et al [2002] for different distance measures
- Applications to Graph Drawing, Video Games, Adhoc Networking etc.

Our Results

Measure	Maximum	Minimum
Sum	O(n log n)	O(n log n)
Product	O(n log n)	O(n log n)
Difference	O(n log n)	$O(n^2 \log n)$
Line-Distance	$O(n^2)$	$O(n^2)$
Triangle Area	O(nh)	$O(n^2)$
Triangle Perimeter	O(nh)	$O(n^2 \log n + \sum_i \sum_j \Phi_i^{\,j})$
Circumradius	$O(n^2 \log n)$	$O(n^2 \log n)$

In the minimum column for the triangle perimeter measure, $\Phi_i^{\,j}$ is a parameter related to point p_i

Sum and Product Measure: Definition

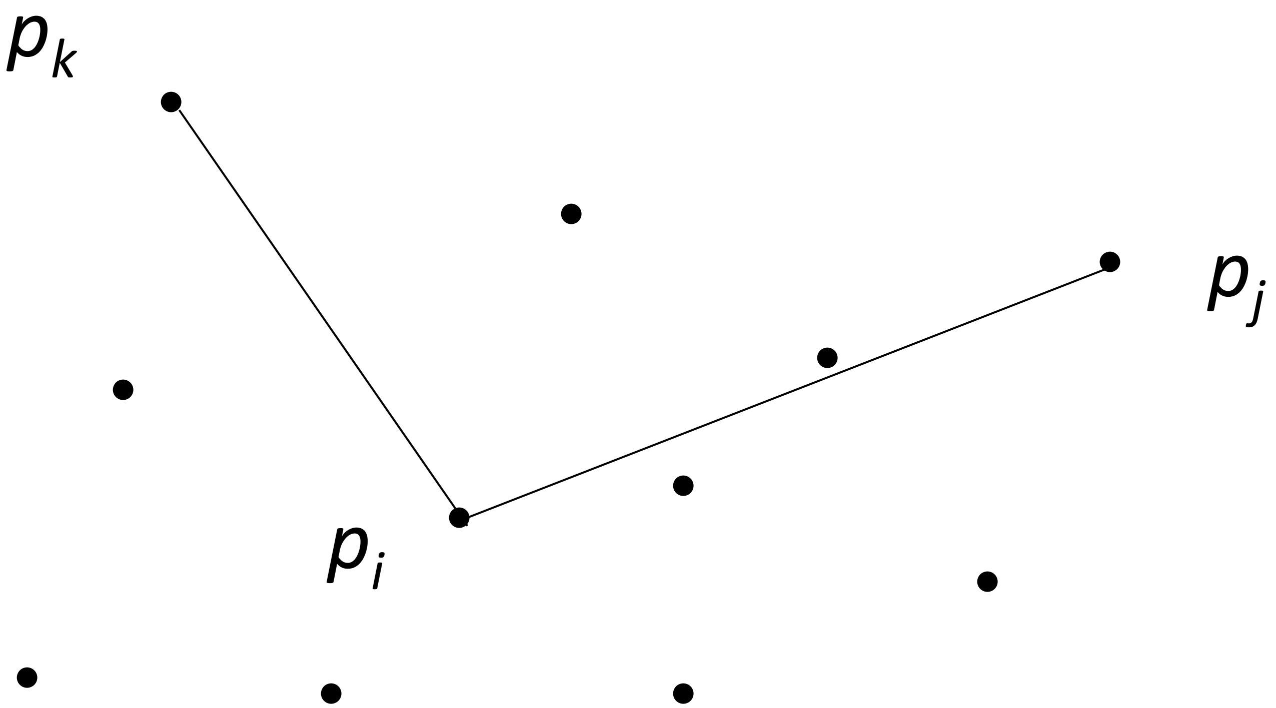
Sum measure $\mathcal{S}(p_i, p_j, p_k) = |\overline{p_i p_j}| + |\overline{p_i p_k}|$

Product measure $\mathcal{P}(p_i, p_j, p_k) = |\overline{p_i p_j}| \times |\overline{p_i p_k}|$

Maximum Sum and Product Measure: Characterization

$\mathcal{S}(p_i, p_j, p_k)$ and $\mathcal{P}(p_i, p_j, p_k)$ is maximum when p_j and $p_k \in P - p_i$, realize the farthest and second farthest distance from point p_i

Maximum Sum and Product Measure : Characterization



Sum and Product Measure: Algorithm

- Construct second- order farthest-point Voronoi diagram
- Build point location structure
- Locate p_i

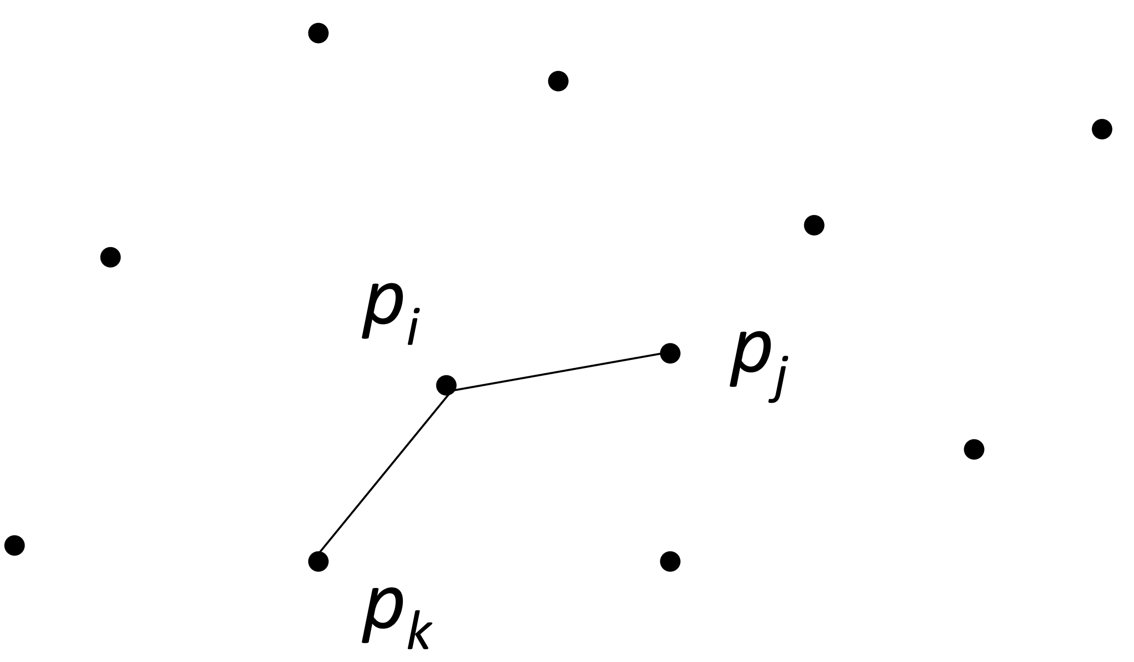
Maximum Sum and Product Measure : Complexity

- Construction of Voronoi diagram : O(n log n)
- Point location over n points : O(n log n)
- Total : O(n log n)
- Lower bound of Ω(n log n) in the algebraic decision tree model by reduction from the all-farthest pairs problem

Minimum Sum and Product Measure : Characterization

$\mathcal{S}(p_i, p_j, p_k)$ and $\mathcal{P}(p_i, p_j, p_k)$ is minimum when p_j and p_k ∈ P – p_i, realize the nearest and second nearest distance from point p_i

Minimum Sum and Product Measure : Characterization



Minimum Sum and Product Measure : Algorithm

- Construct third order nearest-point Voronoi diagram
- Build point location structure
- Locate p_i

Minimum Sum and Product Measure : Complexity

- Construction of Voronoi diagram : O(n log n)
- Point location over n points : O(n log n)
- Thus we have an O(n log n) time algorithm
- Lower bound of Ω(n log n) in the algebraic decision tree model by reduction from the closest pair problem

Difference Measure : Definition

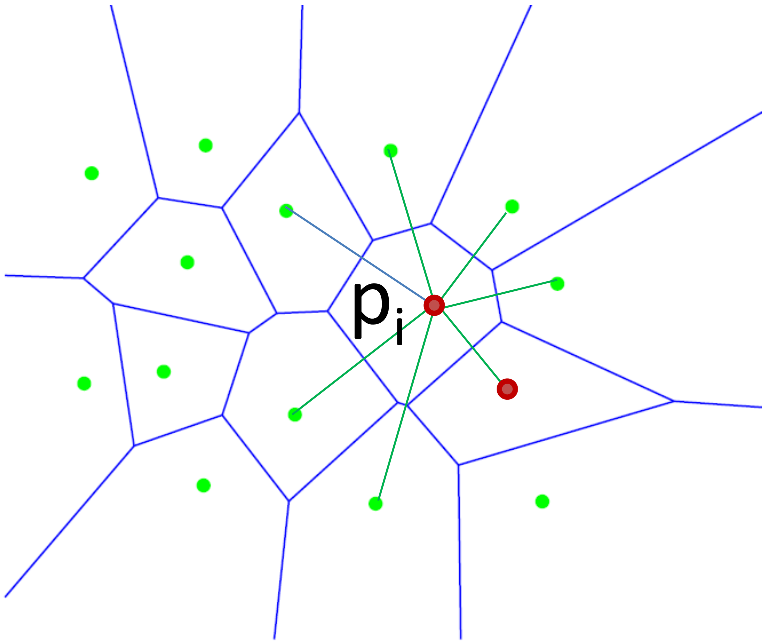
$$\mathcal{D}(p_i, p_j, p_k) = \left|\,|\overline{p_i p_j}| - |\overline{p_i p_k}|\,\right|$$

Maximum Difference Measure : Characterization

For an anchored point p_i, $\mathcal{D}(p_i, p_j, p_k)$ is maximum iff p_j and p_k are respectively the nearest and farthest point from p_i or vice versa

Maximum Difference Measure : Algorithm

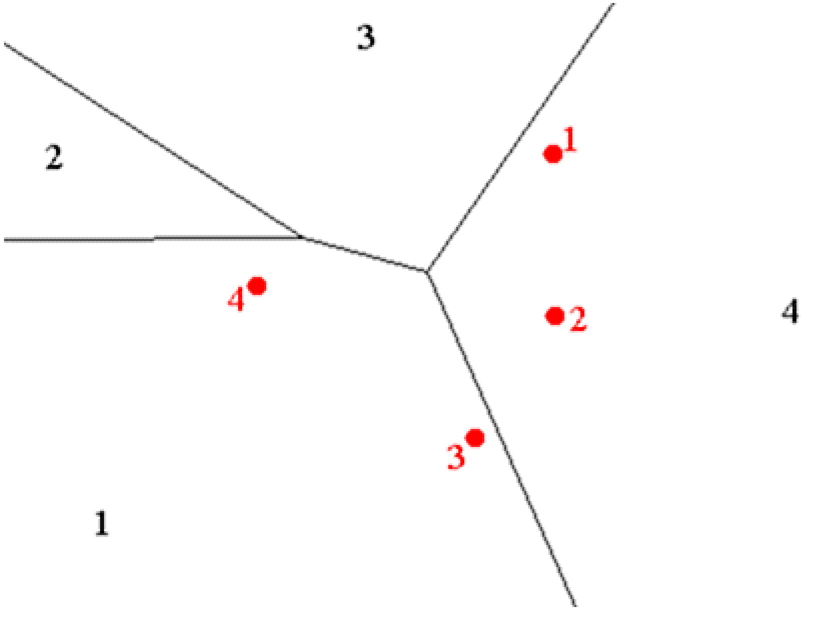
- Find nearest to each p_i from nearest-point Voronoi diagram



Voronoi Diagram

Maximum Difference Measure : Algorithm

- Find farthest point from each p_i using a farthest-point Voronoi diagram and a point location structure

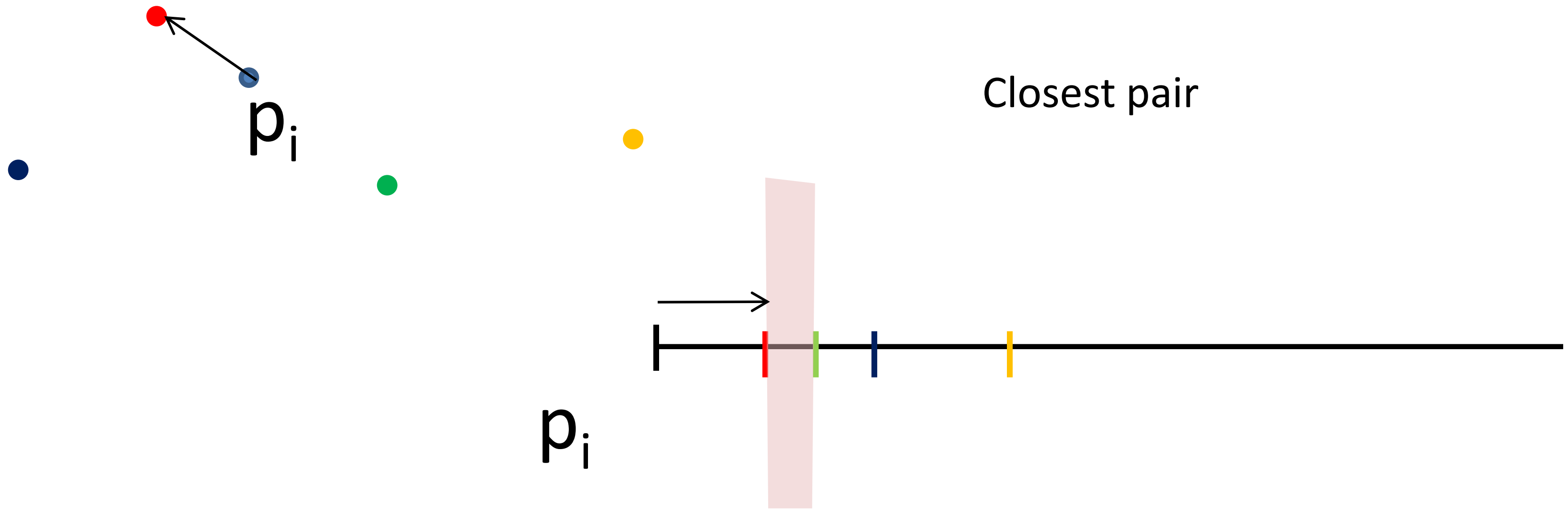


Maximum Difference Measure : Complexity

- Construction of Voronoi diagram : O(n log n)
- Point location over n points : O(n log n)
- Thus we have an O(n log n) time algorithm
- Lower bound of Ω(n log n) in ADT model by reduction from the diameter problem

Minimum Difference Measure : Characterization

- Relative to p_i, problem reduces to finding a closest pair on a line



Minimum Difference Measure : Complexity

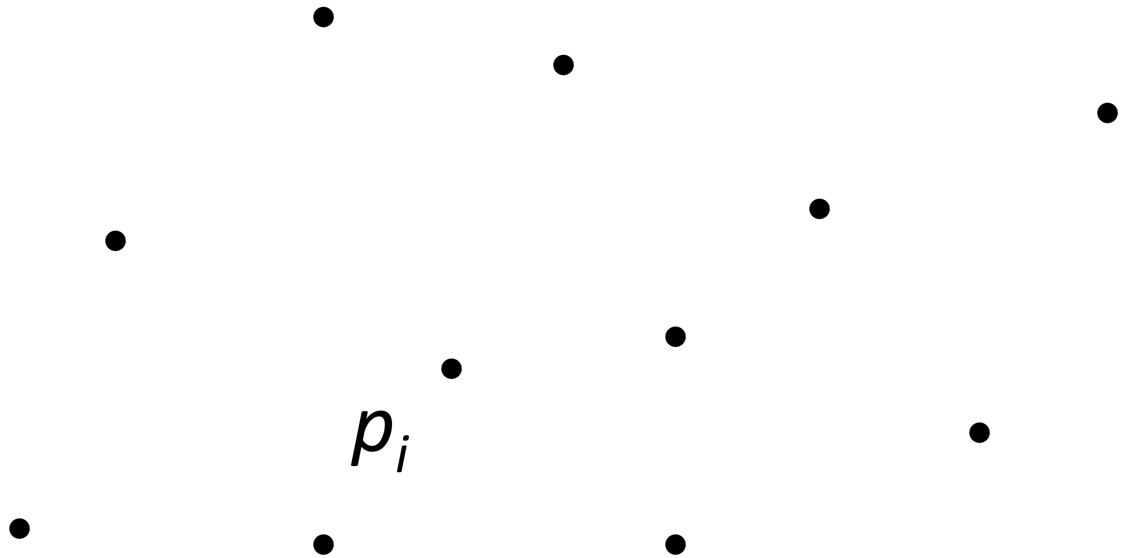
- Closest pair problem for each p_i : $O(n \log n)$
- Over n points : $O(n^2 \log n)$
- Lower bound of $\Omega(n \log n)$ in the ADT model by reduction from the closest pair problem
- $O(n^2)$ time when the points lie on a line

Line Distance Measure : Definition

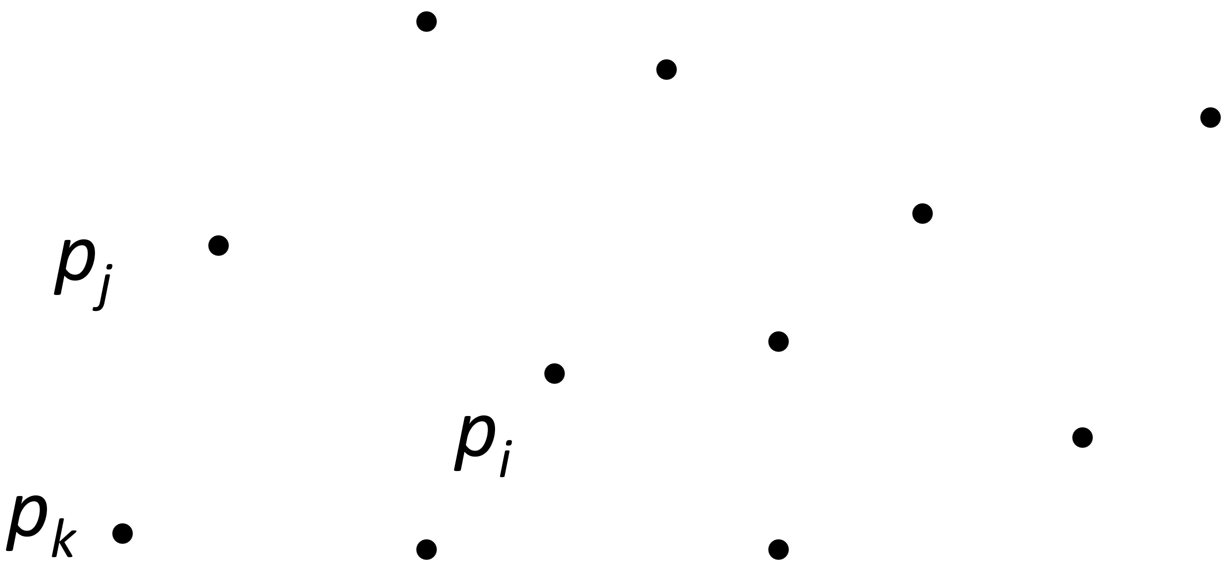
$$\mathcal{LD}(p_i, p_j, p_k) = d(p_i, \overleftrightarrow{p_j p_k})$$

$d(p, \ell)$ is the distance of point p from line ℓ

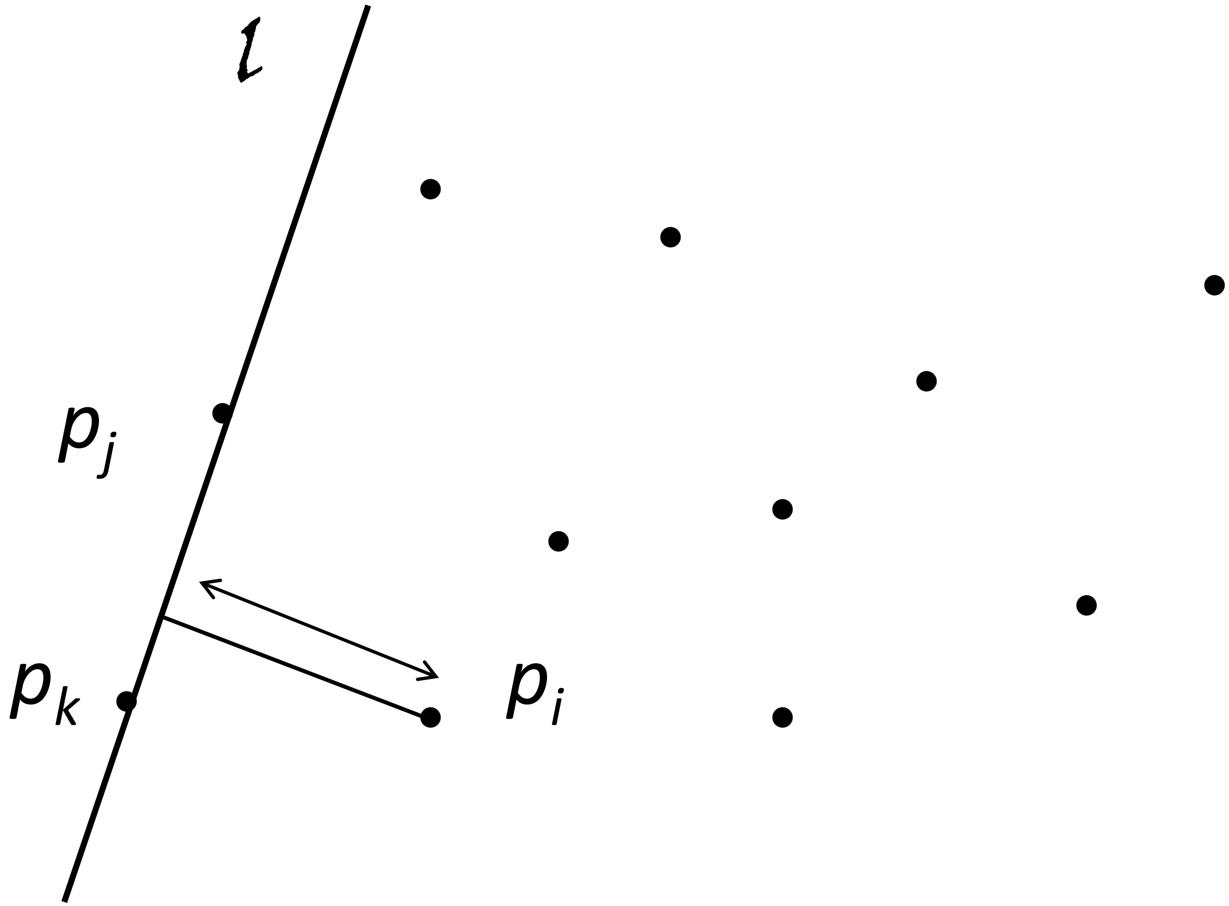
Line Distance Measure: Definition



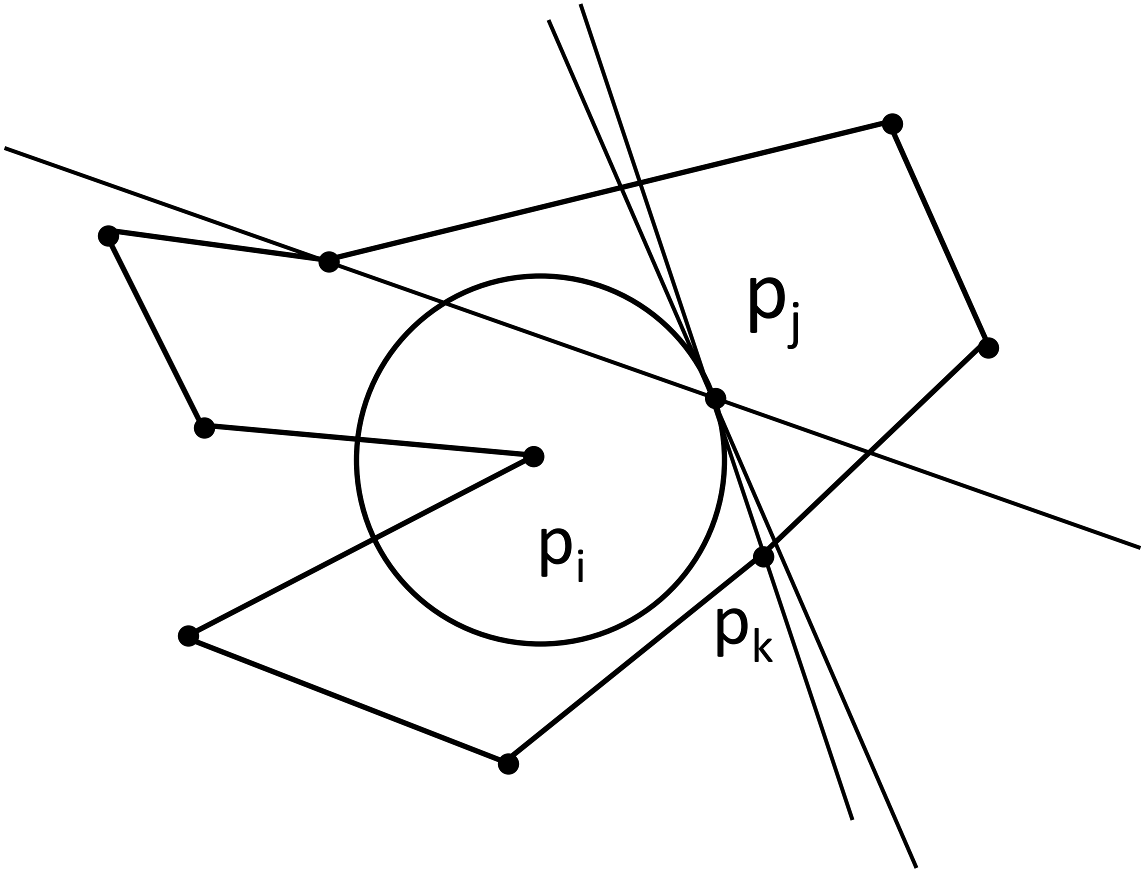
Line Distance Measure: Definition



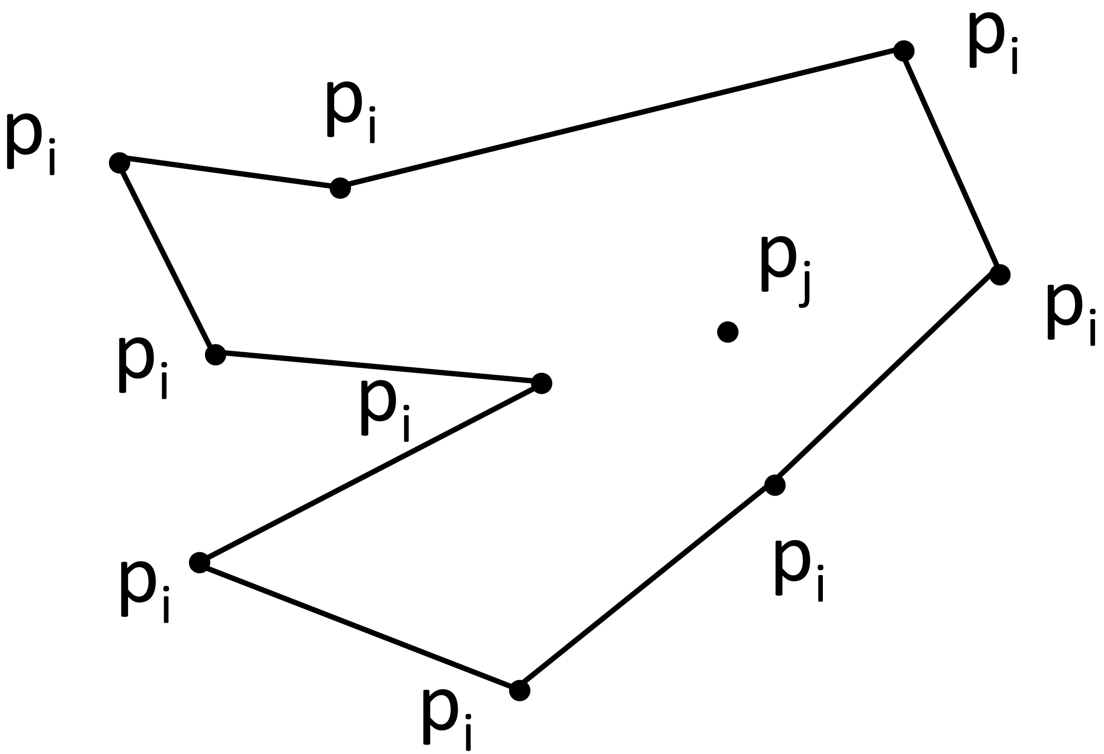
Line Distance Measure: Definition



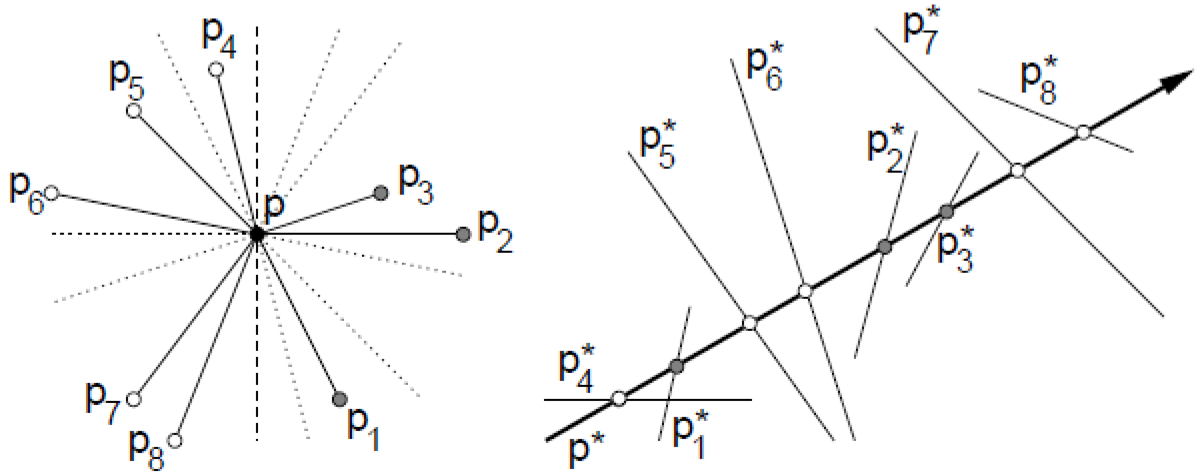
Maximum Line Distance Measure : Characterization



Maximum Line Distance Measure: Algorithm



Maximum Line Distance Measure: Algorithm



Angular sequence of points around point P

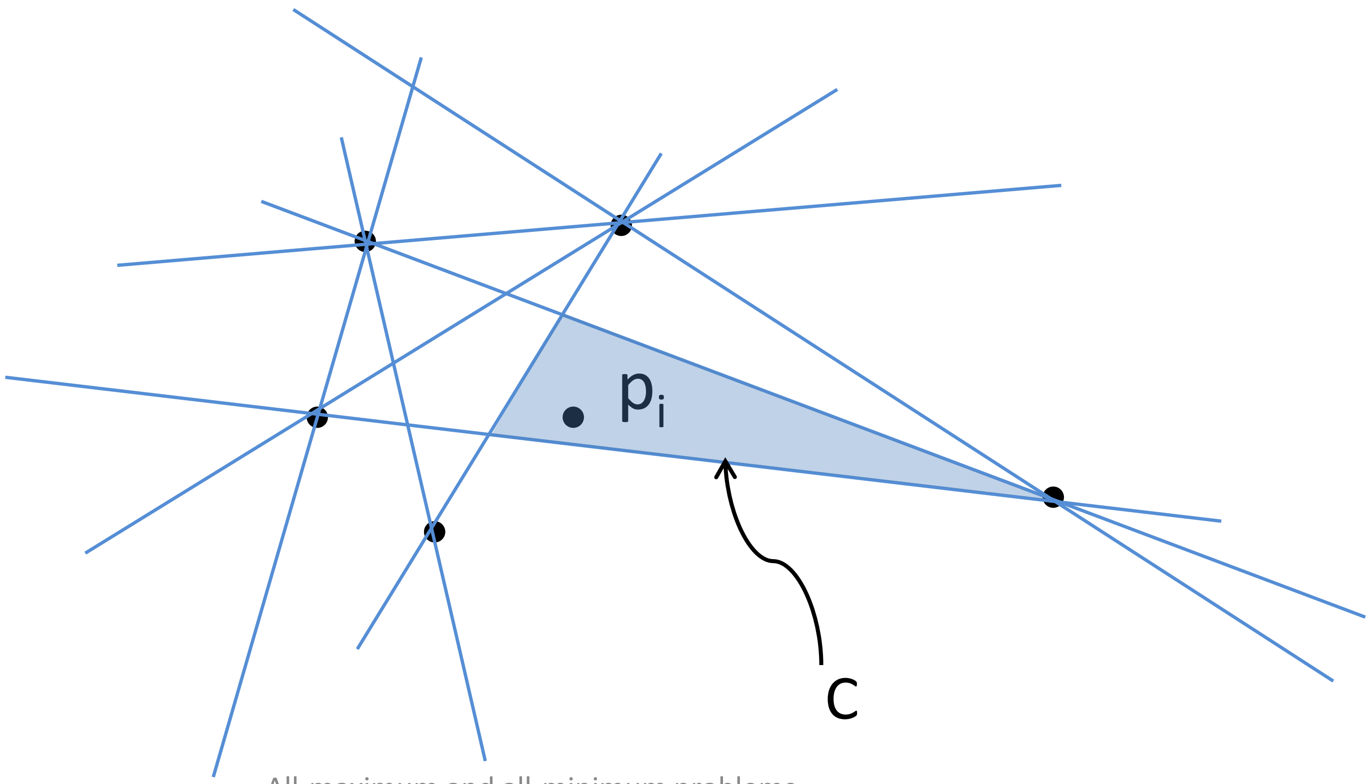
Courtesy:
Mount's Note
http://www.cs.umd.edu/~mount/754/Lects/754lects.pdf

Maximum Line Distance Measure : Complexity

- Angular order about all p_j : $O(n^2)$
- Farthest line thru’ p_j for all p_i : $O(n)$
- Farthest line from each p_i in P : $O(n^2)$
- Total time complexity : $O(n^2)$

Minimum Line Distance Measure : Characterization

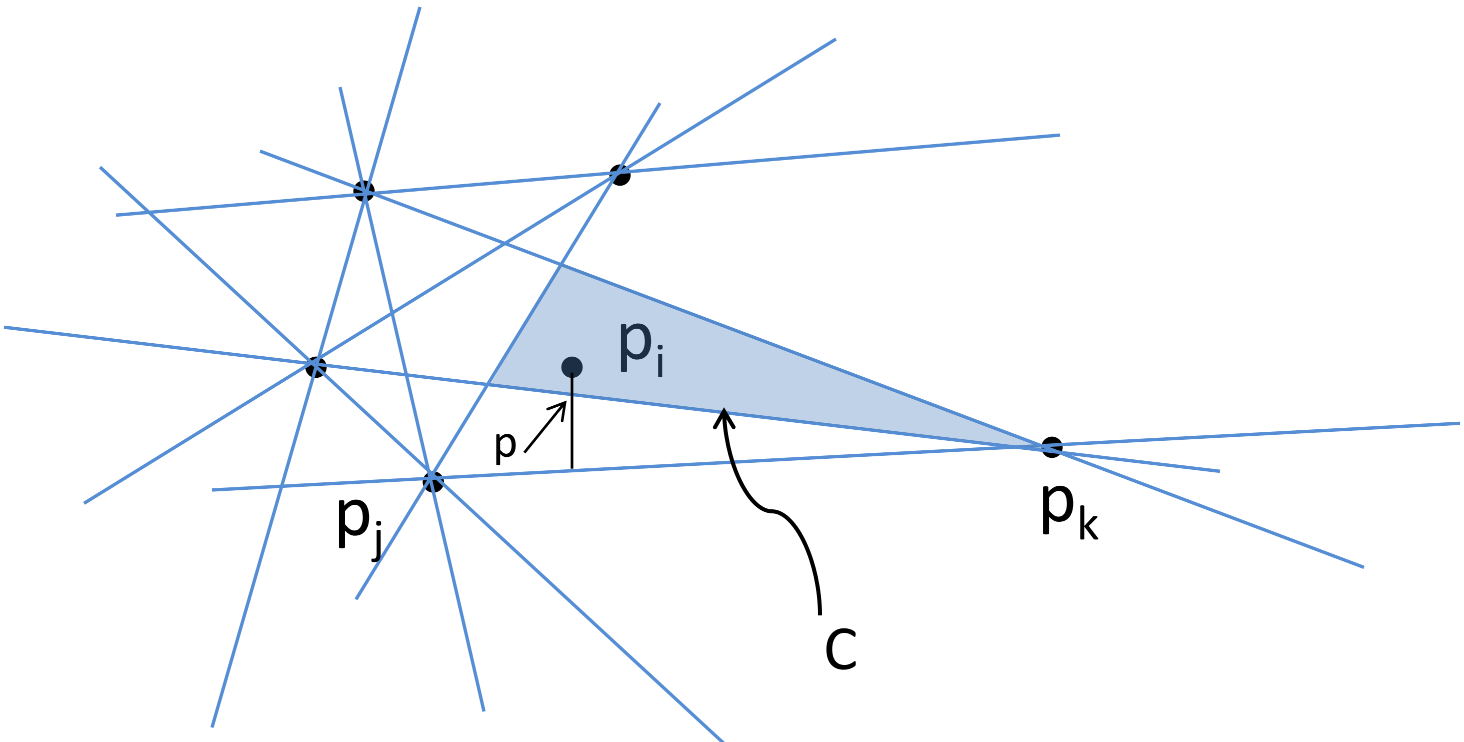
- Arrangement of lines from all pairs in $P - \{p_i\}$



Minimum Line Distance Measure: Characterization

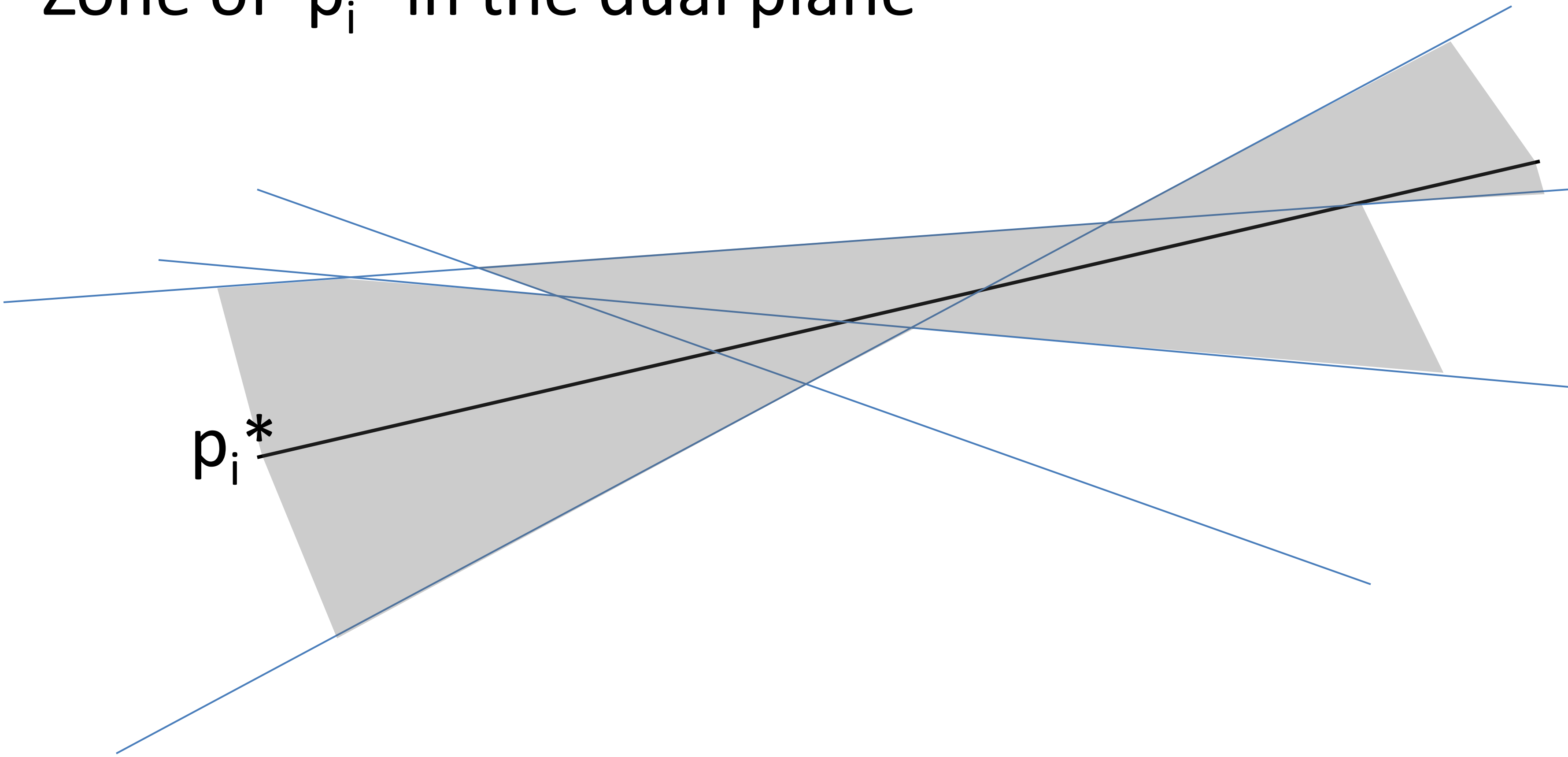
- Line closest to p_i is a bounding line of cell C

Proof :

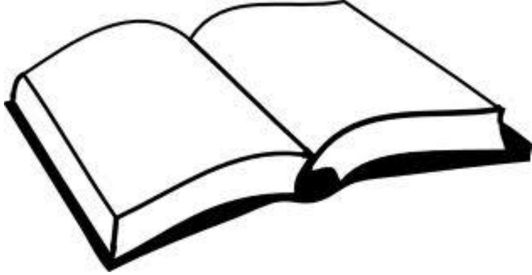


Minimum Line Distance Measure: Algorithm

- Zone of p_i^* in the dual plane



Duality



Point- Line duality:

Maps points (lines) in primal plane to lines (points) in the dual plane

xy plane(primal plane)

p: (p_x, p_y)

l: $y = l_u . x - l_v$

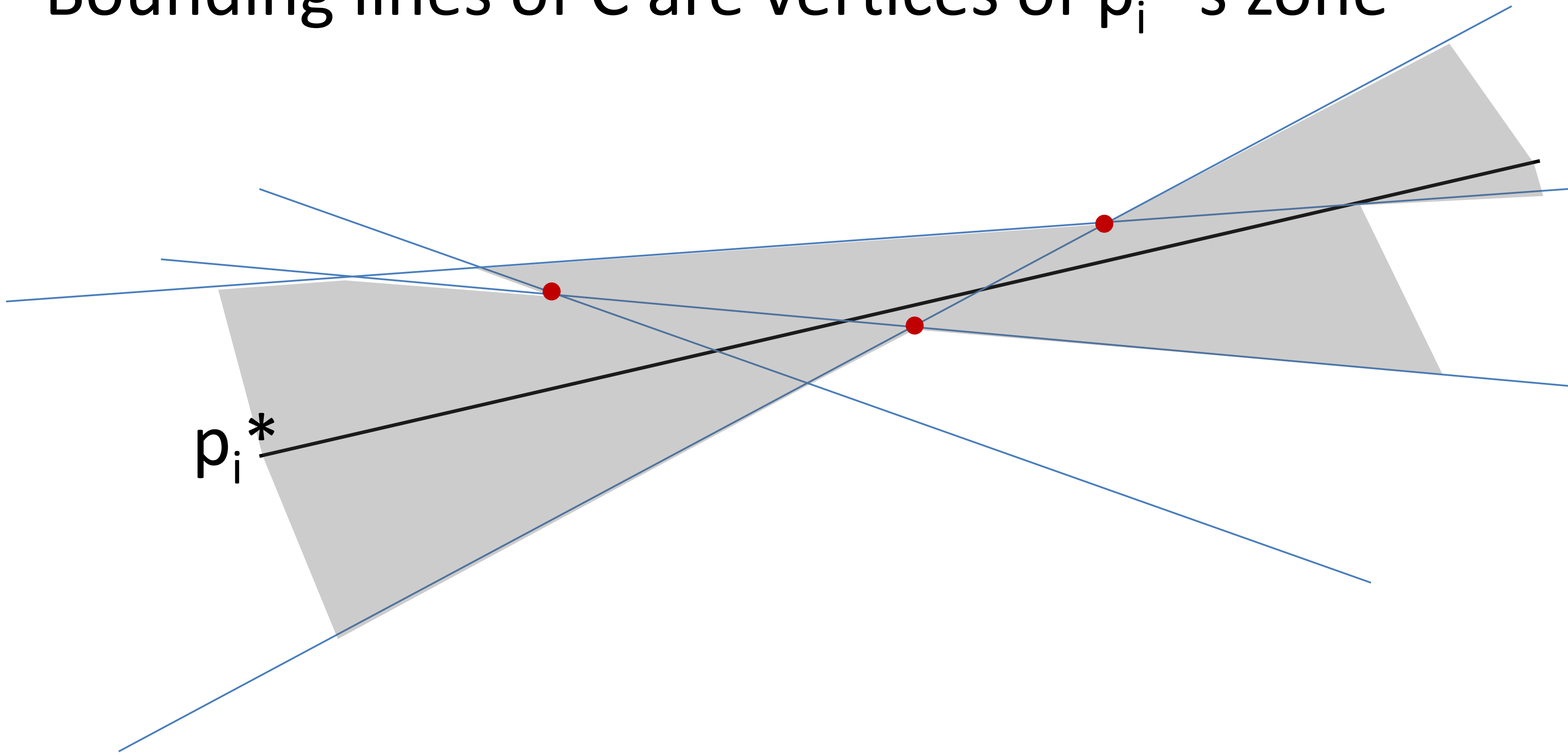
uv plane(dual plane)

p* : $v = p_x.u - p_y$

l*: (l_u, l_v)

Minimum Line Distance Measure: Algorithm

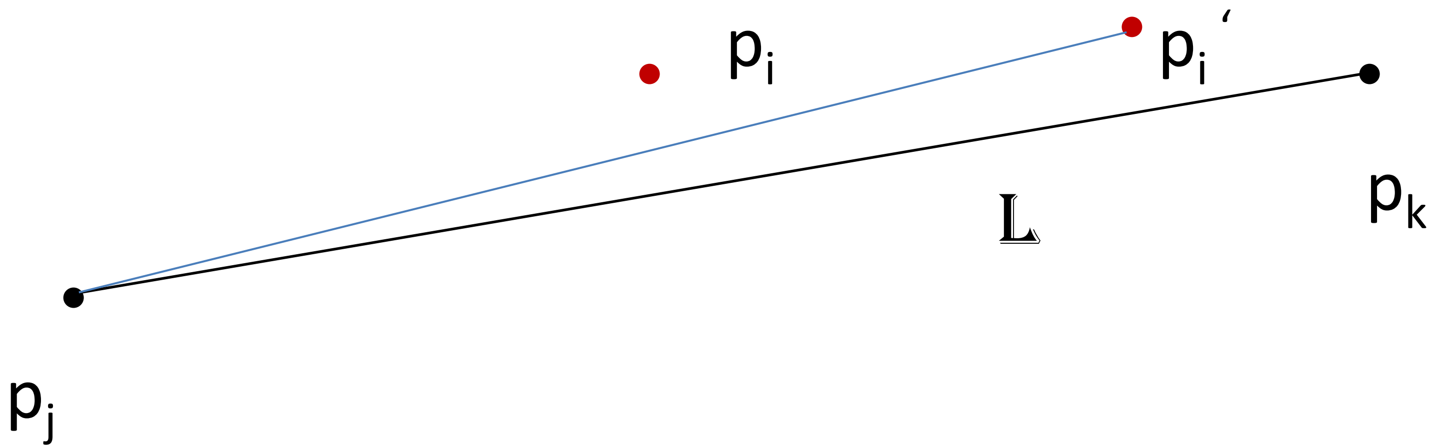
- Bounding lines of C are vertices of p_i*'s zone



Minimum Line Distance Measure: Characterization

Proof :

Contradiction !



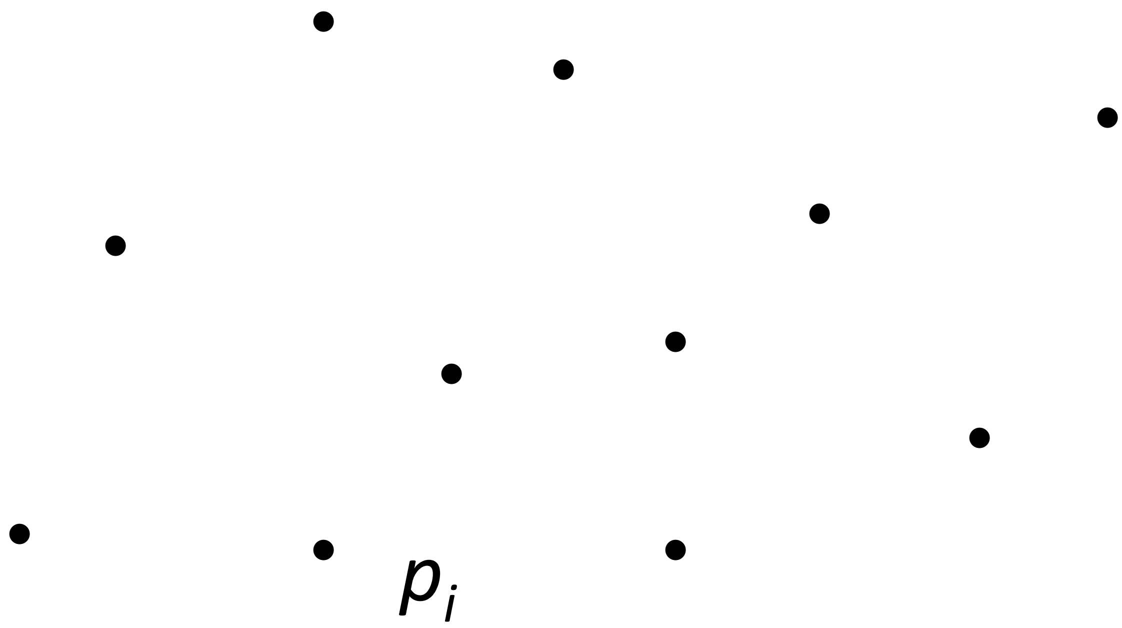
Minimum Line Distance Measure : Complexity

- Construction of arrangement : $O(n^2)$
- Closest line to p_i from zone of p_i^*: $O(n)$
- Over n points : $O(n^2)$
- Problem is n^2-hard by reduction from the problem of determining if 3 of n points in the plane are collinear

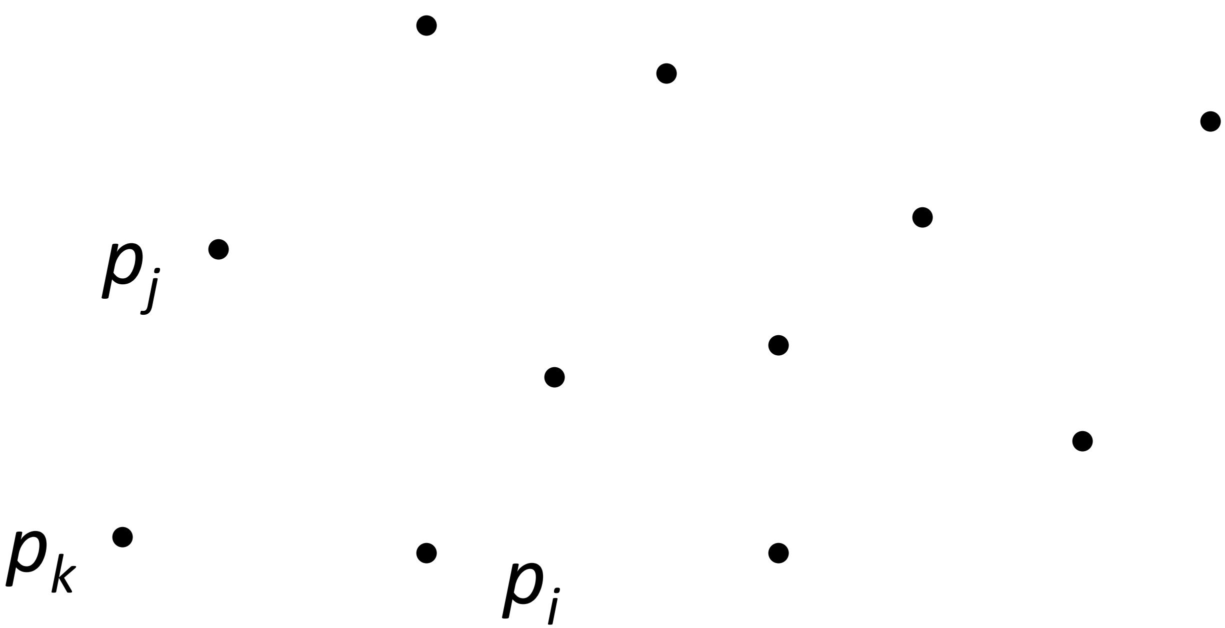
Triangle Area Measure: Definition

$$\mathcal{A}(p_i, p_j, p_k) = \text{Area of the } \Delta p_i p_j p_k$$

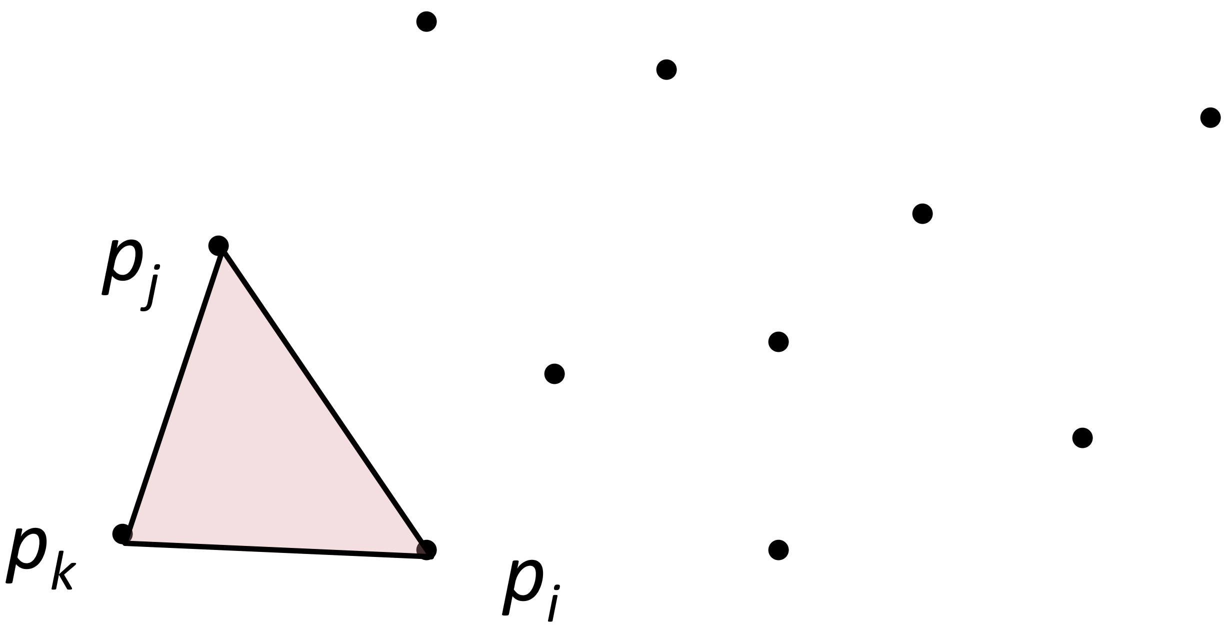
Triangle Area Measure: Definition



Triangle Area Measure: Definition



Triangle Area Measure : Definition

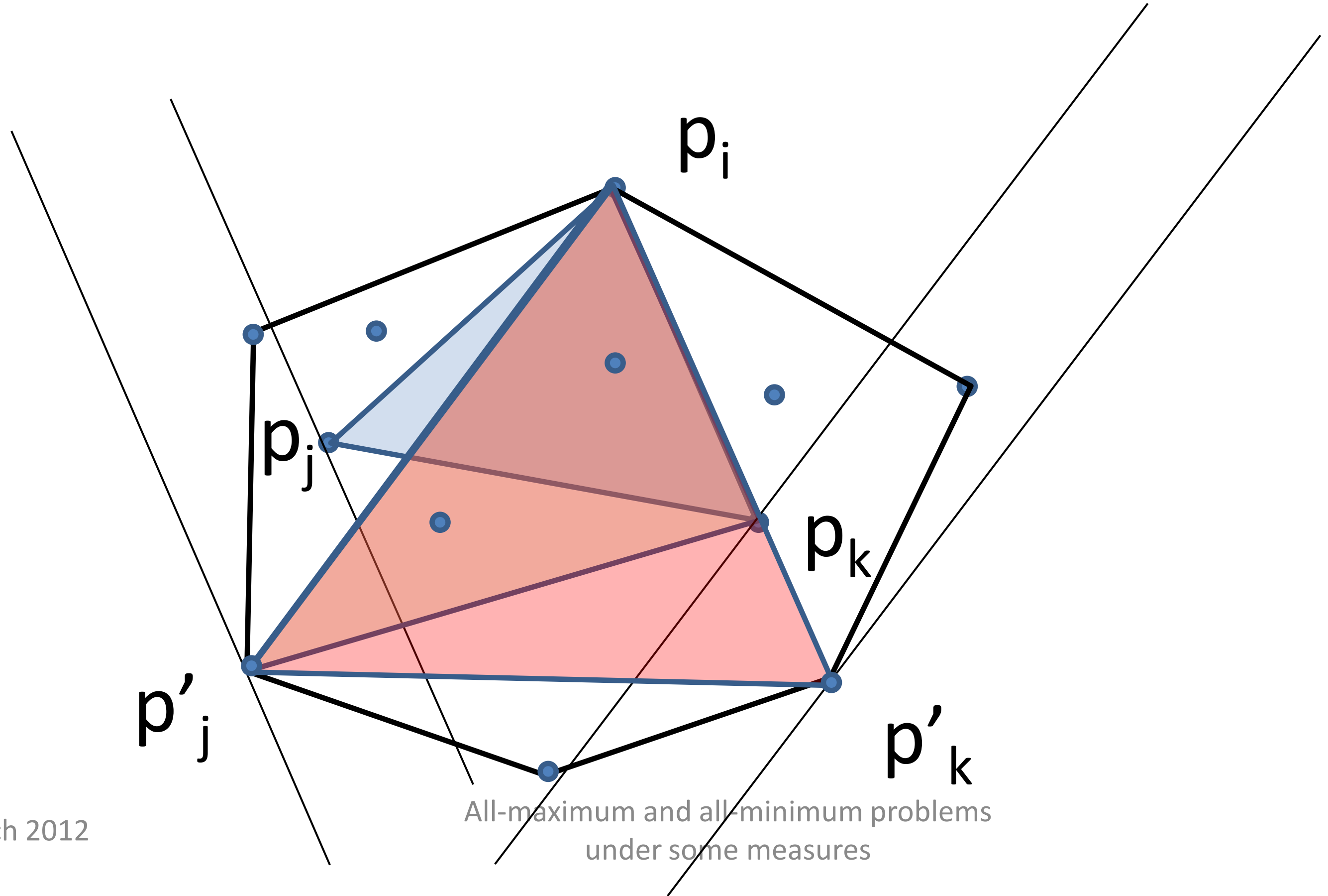


Maximum Area Triangle Measure: Characterization

- For a point p_i, if $\mathcal{A}(p_i, p_j, p_k)$ is maximum then p_j and p_k are points on the convex hull of P

Maximum Area Triangle Measure: Characterization

Proof :



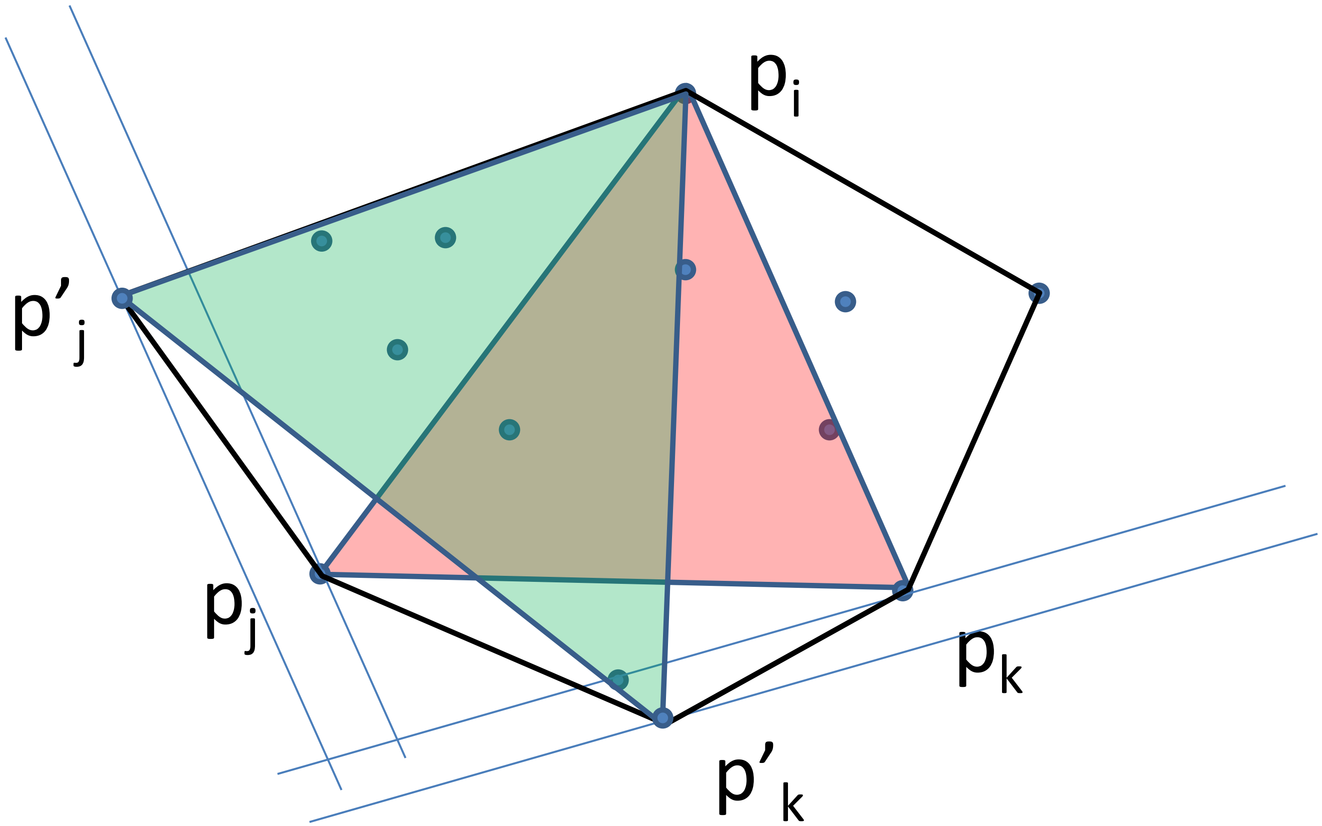
Contradiction !

Maximum Area Triangle Measure: Characterization

- For a point p_i, if $\mathcal{A}(p_i, p_j, p_k)$ is maximum for a pair $\{p_j, p_k\}$, then p_j is the farthest point from the supporting line of p_ip_k and p_k is the farthest point from the supporting line of p_ip_j

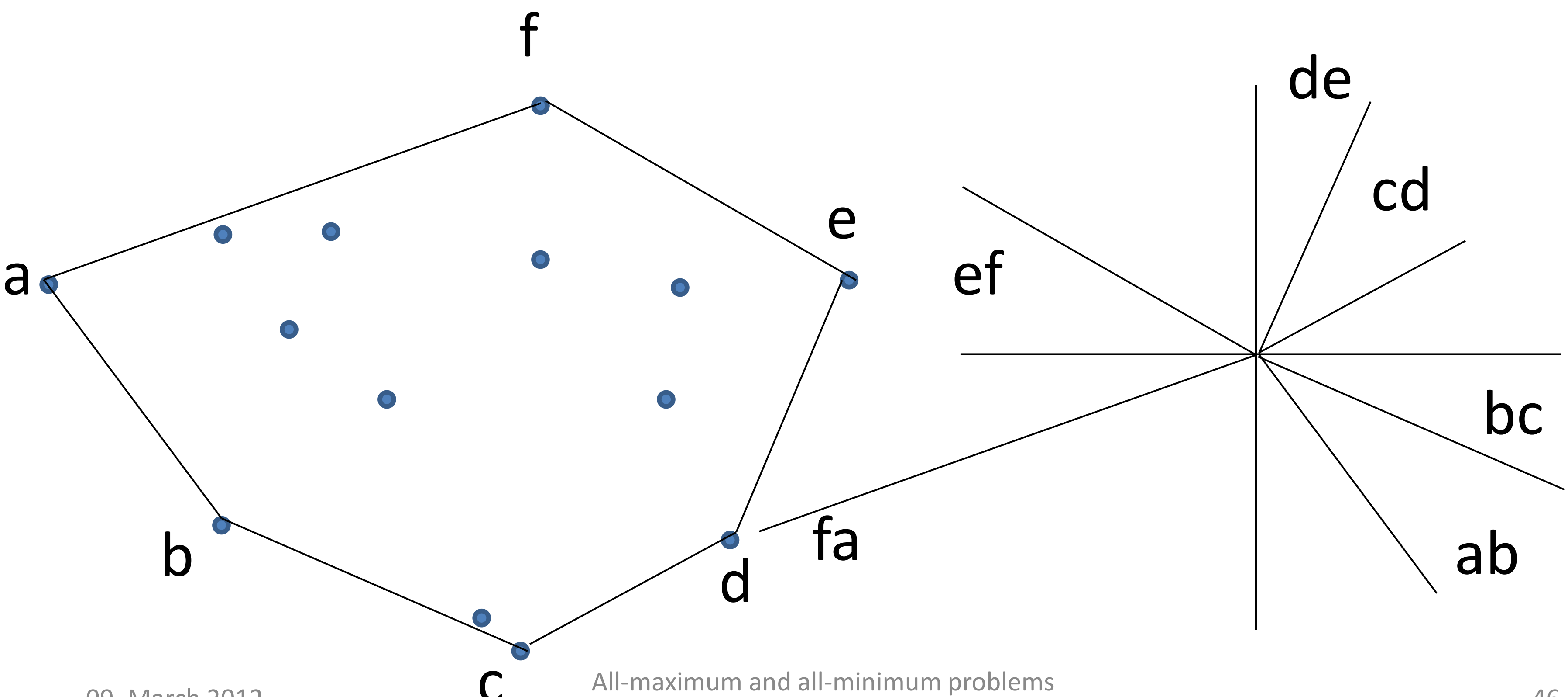
Maximum Area Triangle Measure: Characterization

Proof :



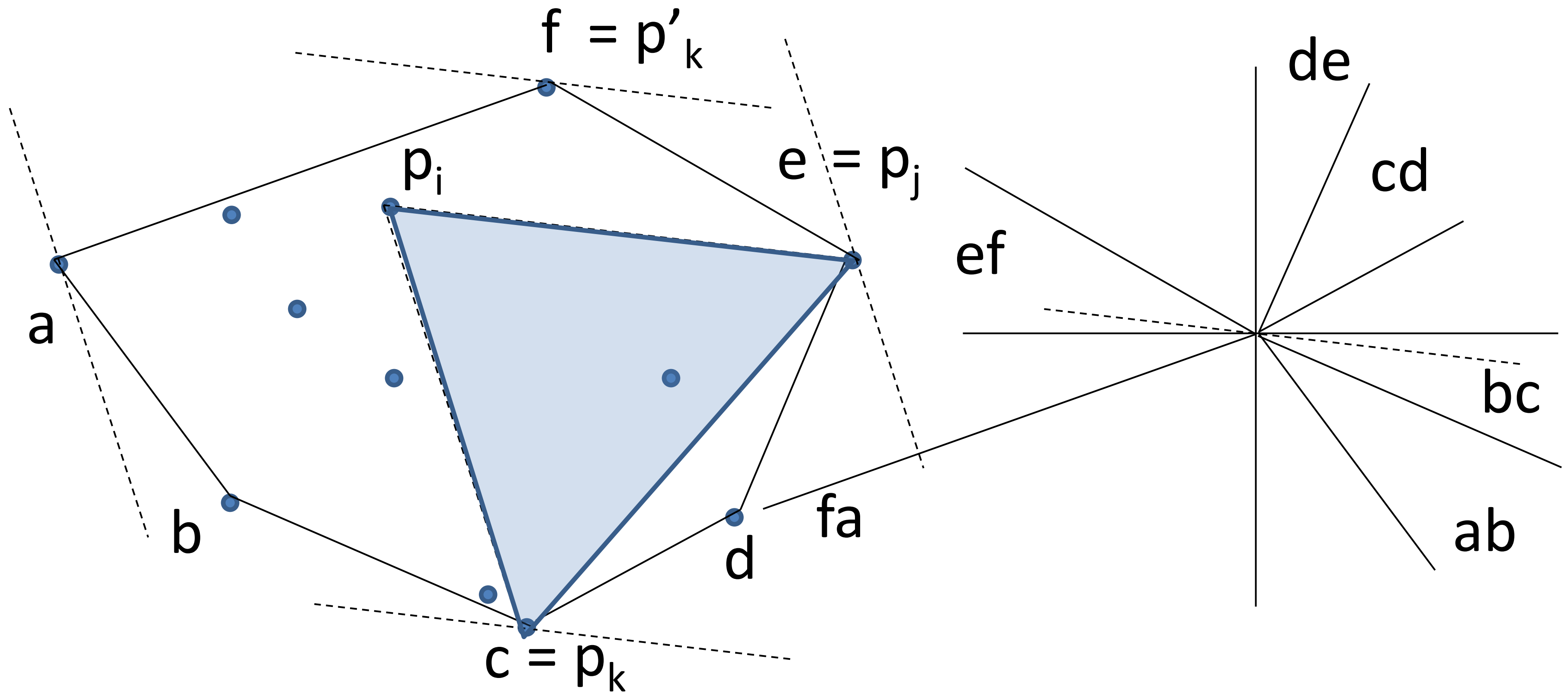
Maximum Area Triangle Measure: Algorithm

- **Preprocessing Step** : Construct the convex hull of P and then its ray diagram



Maximum Area Triangle Measure : Algorithm

- For each p_i we scan the boundary of convex hull boundary for p_j and p_k



Maximum Area Triangle Measure: Complexity

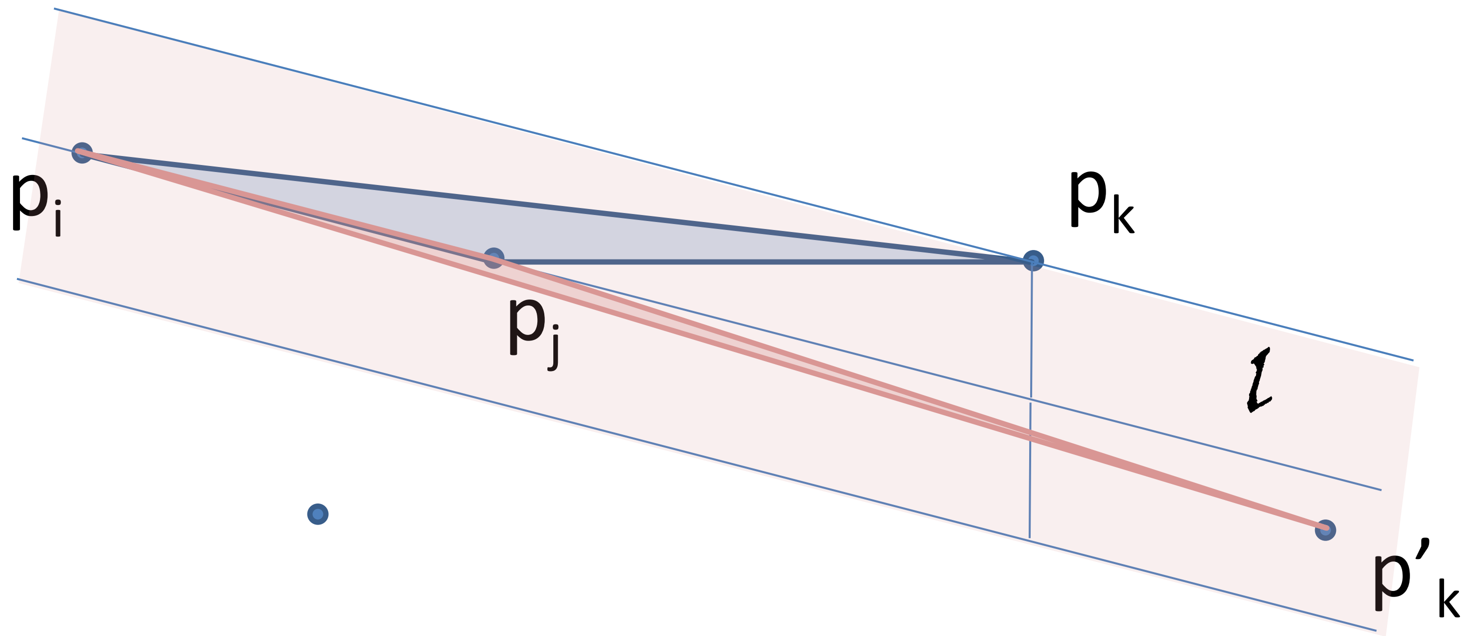
- Preprocessing Step : O (n log h)
- Maximum area triangle rooted at p_i : O(h)
- Over n points : O(nh)

Minimum Area Triangle Measure: Characterization

For an anchored point p_i and a fixed p_j, if $\mathcal{A}(p_i, p_j, p_k)$ is minimum then p_k is vertically closest to the supporting line, l , of p_i and p_j

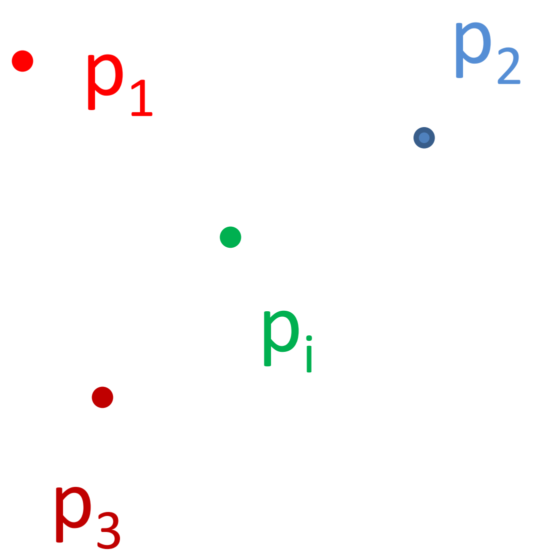
Minimum Area Triangle Measure: Characterization

Proof :

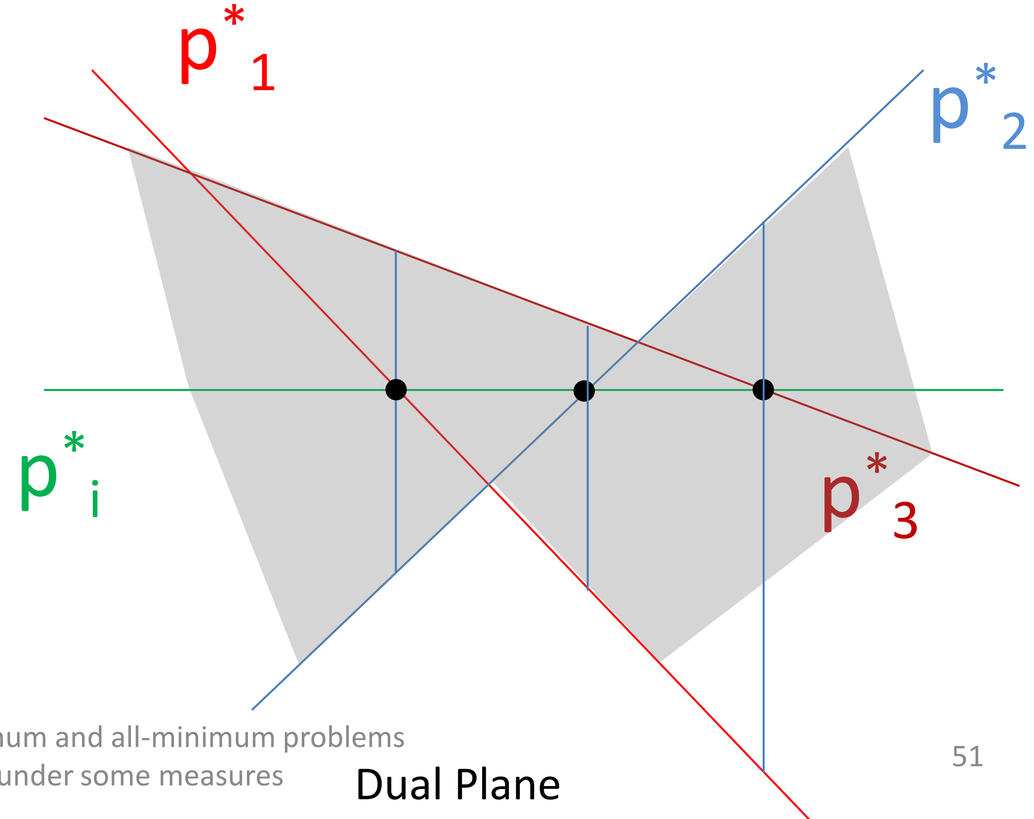


Minimum Area Triangle Measure : Algorithm

- The vertically closest line to each intersection point on p^*_i is part of the zone of p_i^*



Primal Plane



Dual Plane

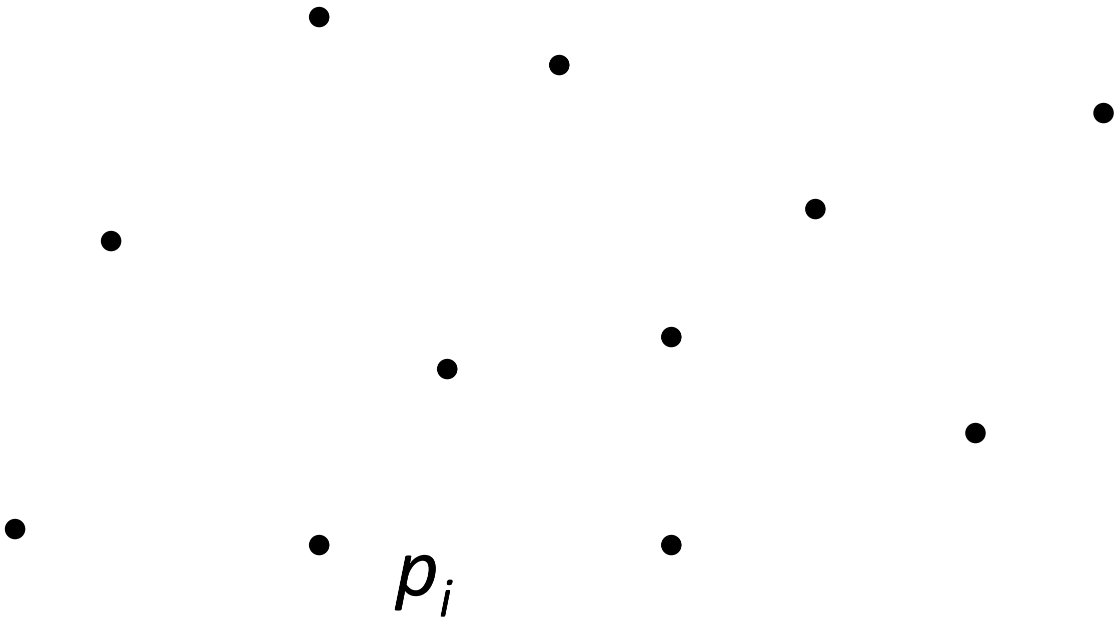
Minimum Area Triangle Measure : Complexity

- Construction of arrangement : $O(n^2)$
- Minimum area triangle rooted at p_i from zone of p_i^*: $O(n)$
- Over n points : $O(n^2)$
- This problem is n^2-hard by reduction from the problem of determining if 3 of n points in the plane are collinear

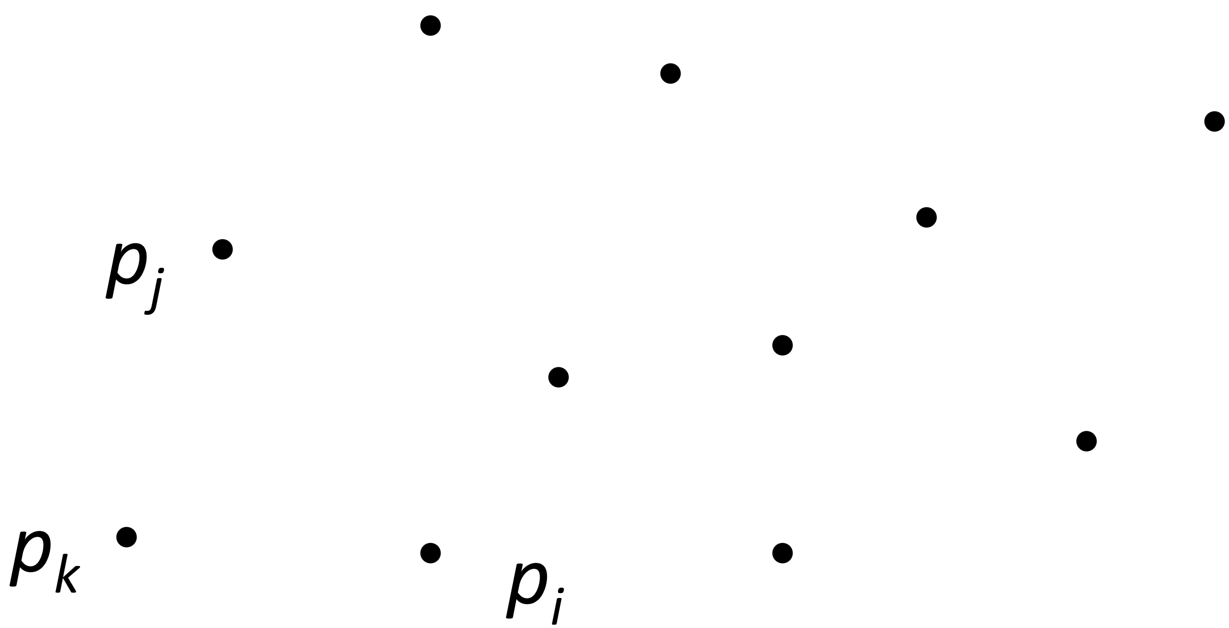
Triangle Perimeter Measure: Definition

$$\mathcal{P}(p_i, p_j, p_k) = |\overline{p_i p_j}| + |\overline{p_j p_k}| + |\overline{p_k p_i}|$$

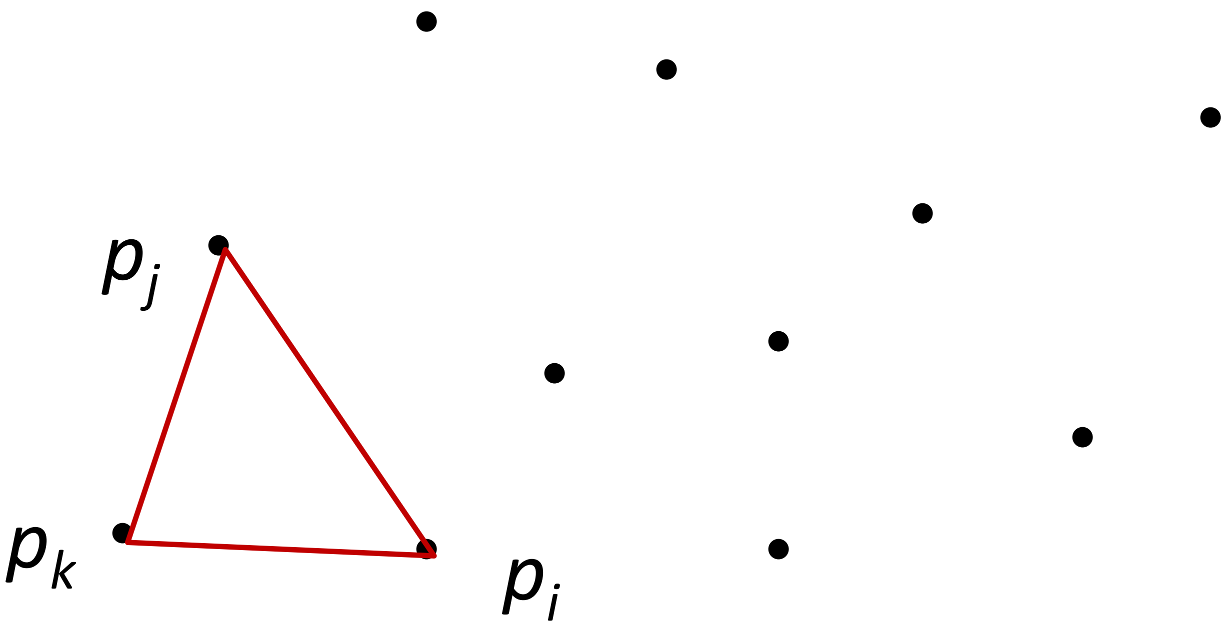
Triangle Perimeter Measure: Definition



Triangle Perimeter Measure: Definition



Triangle Perimeter Measure: Definition

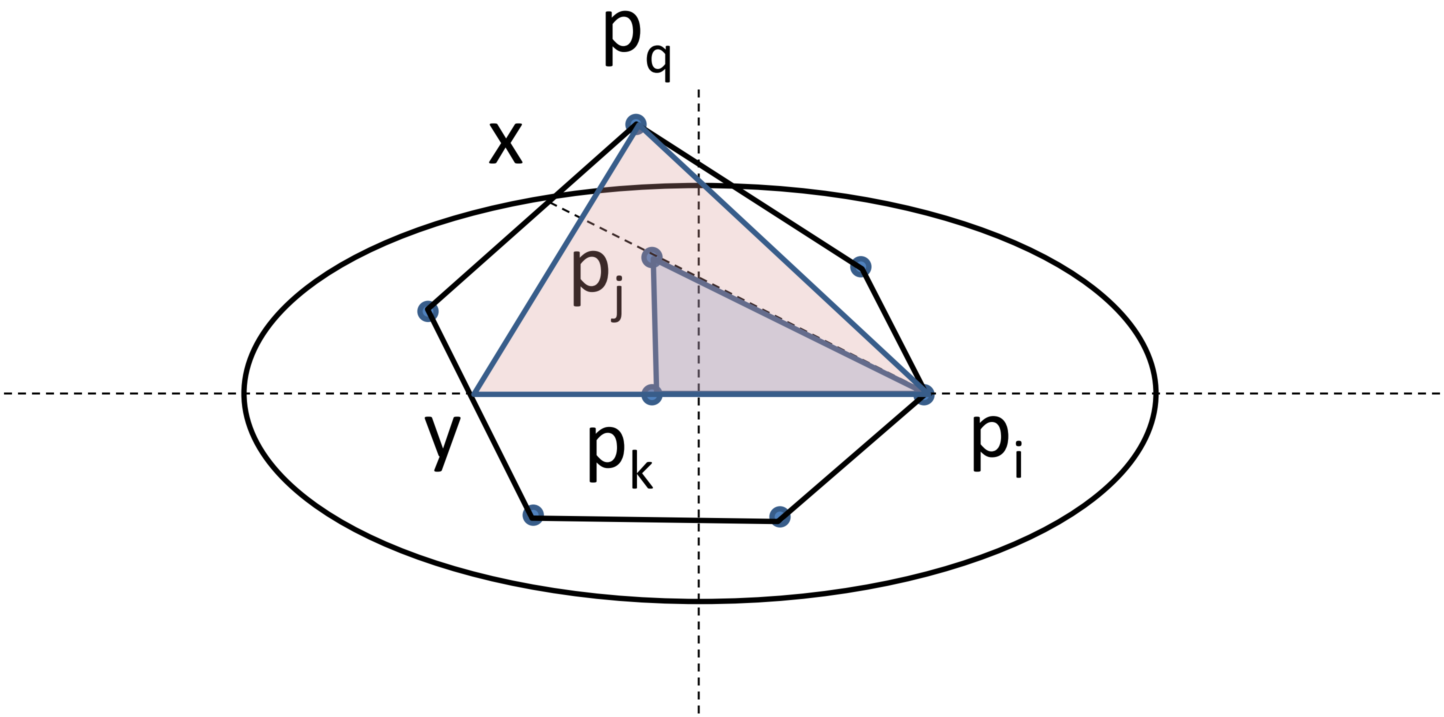


Maximum Triangle Perimeter Measure: Characterization

For a point $p_i \in P$ if the perimeter $\mathcal{P}(p_i, p_j, p_k)$ is maximum then the pair $\{p_j, p_k\} \in P - \{p_i\}$ lie on the convex hull, CH(P), of P.

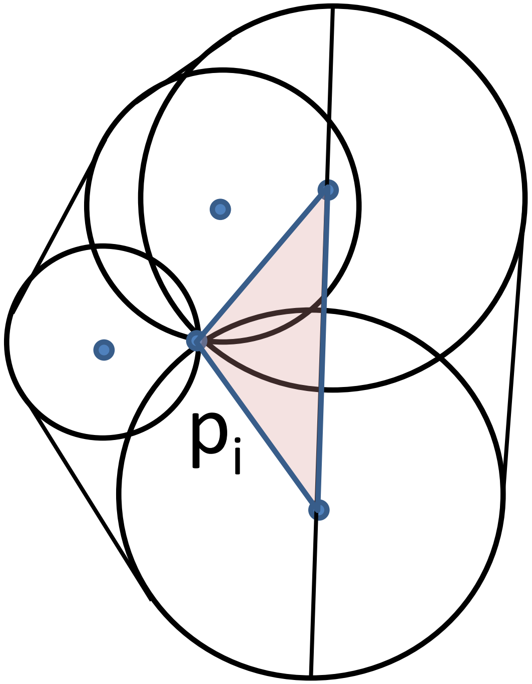
Maximum Triangle Perimeter Measure : Characterization

Proof :



Maximum Triangle Perimeter Measure: Algorithm

- Maximum perimeter triangle rooted at p_i (internal to CH(P)) reduces to computing the diameter of a convex figure bounded by circular arcs and tangents to pair of circles [Boyce et al 1985]



Maximum Triangle Perimeter Measure : Complexity

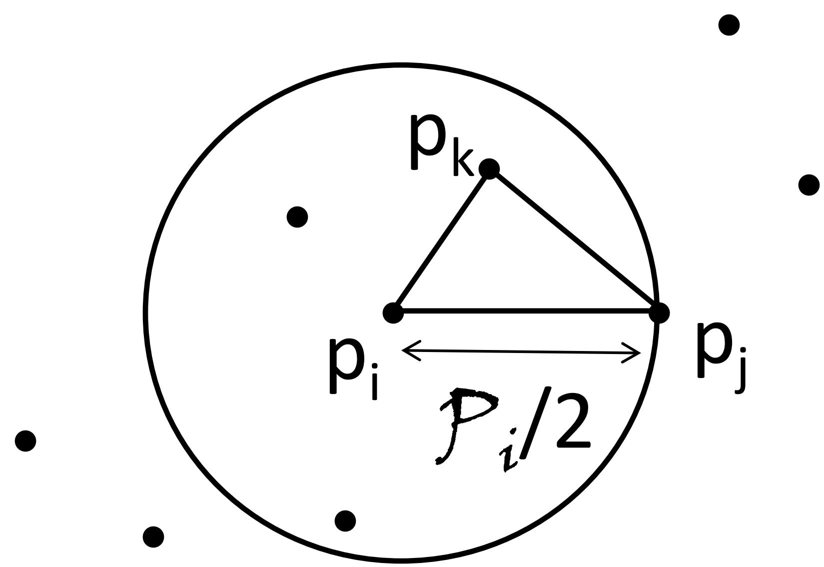
- For all p_i's on CH(P) by the Monotone Matrix method [Aggarwal et al. 1988] : O(h log h)
- For points internal to CH(P) by Boyce's method : O((n-h)h)
- Thus an O(nh) algorithm

Minimum Triangle Perimeter Measure: Characterization

If $\mathcal{P}_i$ is the perimeter of any triangle $\Delta p_i p_j p_k$, anchored at p_i, then both p_j and p_k is at a distance less than $\mathcal{P}_i/2$ from p_i

Minimum Triangle Perimeter Measure: Characterization

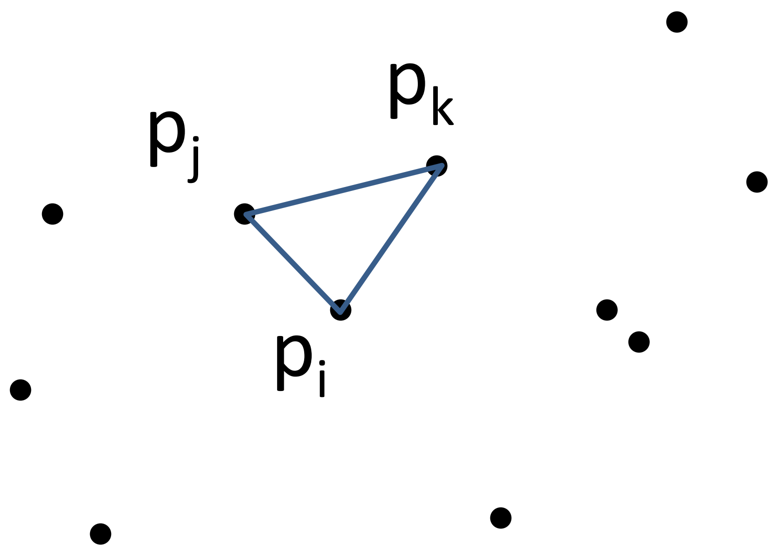
Proof :



Minimum Triangle Perimeter Measure: Algorithm

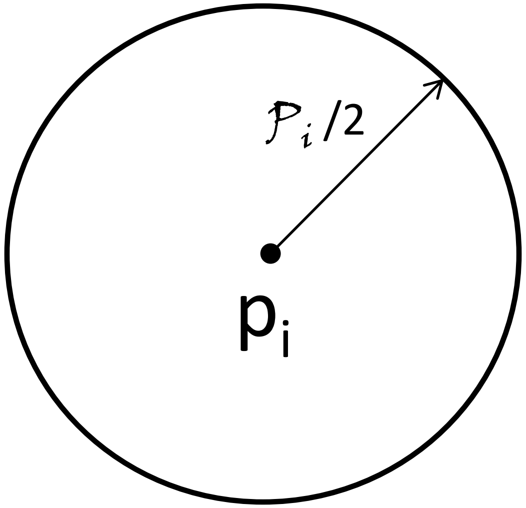
- Initialize $\mathcal{P}_i$

p_j, p_k are closest to p_i



Minimum Triangle Perimeter Measure: Algorithm

- Check if there exists p_j, p_k inside circle such that perimeter of $\Delta(p_i,p_j,p_k) < \mathcal{P}_i$



Minimum Triangle Perimeter Measure: Algorithm

- **YES**: reset $\mathcal{P}_i$ and repeat the last 2 steps
- **NO**: Pick another p_i and continue

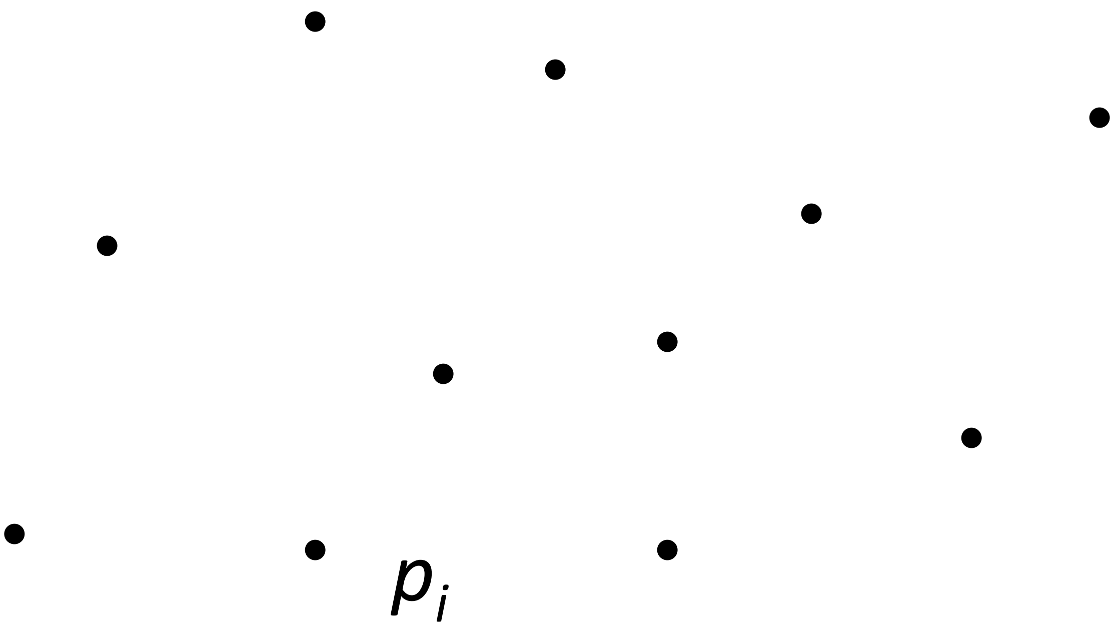
Minimum Triangle Perimeter Measure : Complexity

- Upper bound on the number of points inside a circle of radius $\mathcal{P}_i/2$ is $\mathcal{P}_i/2\Delta_i$, where Δ_i is the smallest separation of pair of adjacent distances
- Determining Δ_i for each p_i : $O(n \log n)$
- Over all p_i : $O(n^2 \log n + \sum_i \sum_j (\mathcal{P}_i^{\,j}/2\Delta_i)^2)$

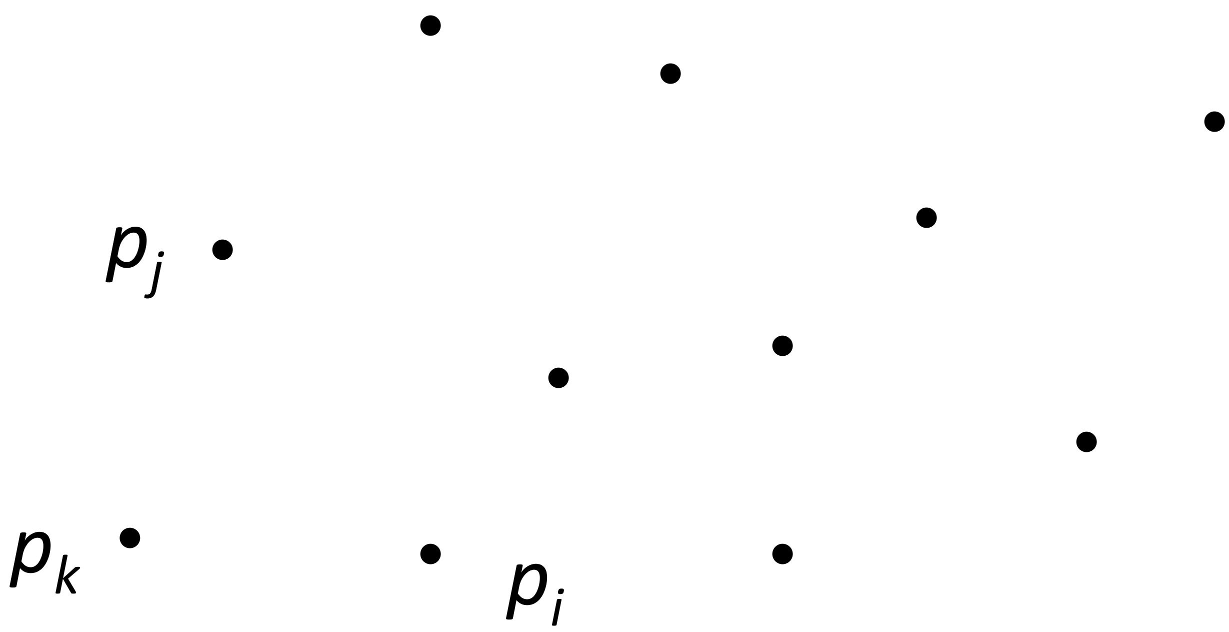
Circumcircle radius Measure : Definition

$\mathcal{R}(p_i, p_j, p_k)$ = Radius of the circle that circumscribes $\Delta p_i p_j p_k$

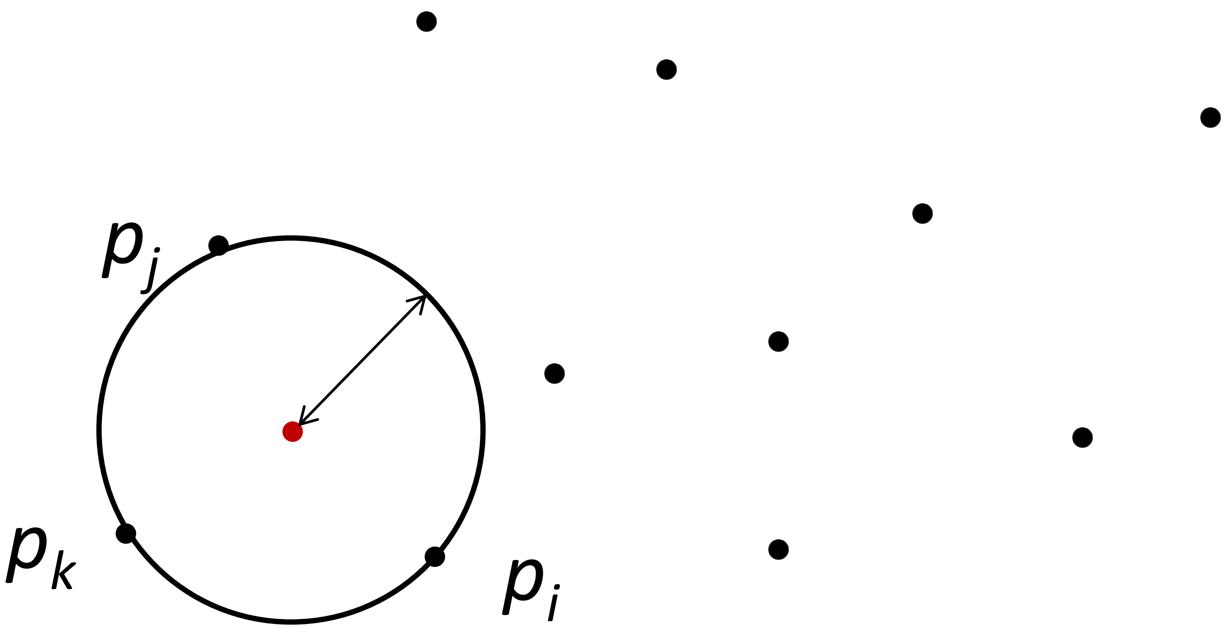
Circumcircle radius Measure: Definition



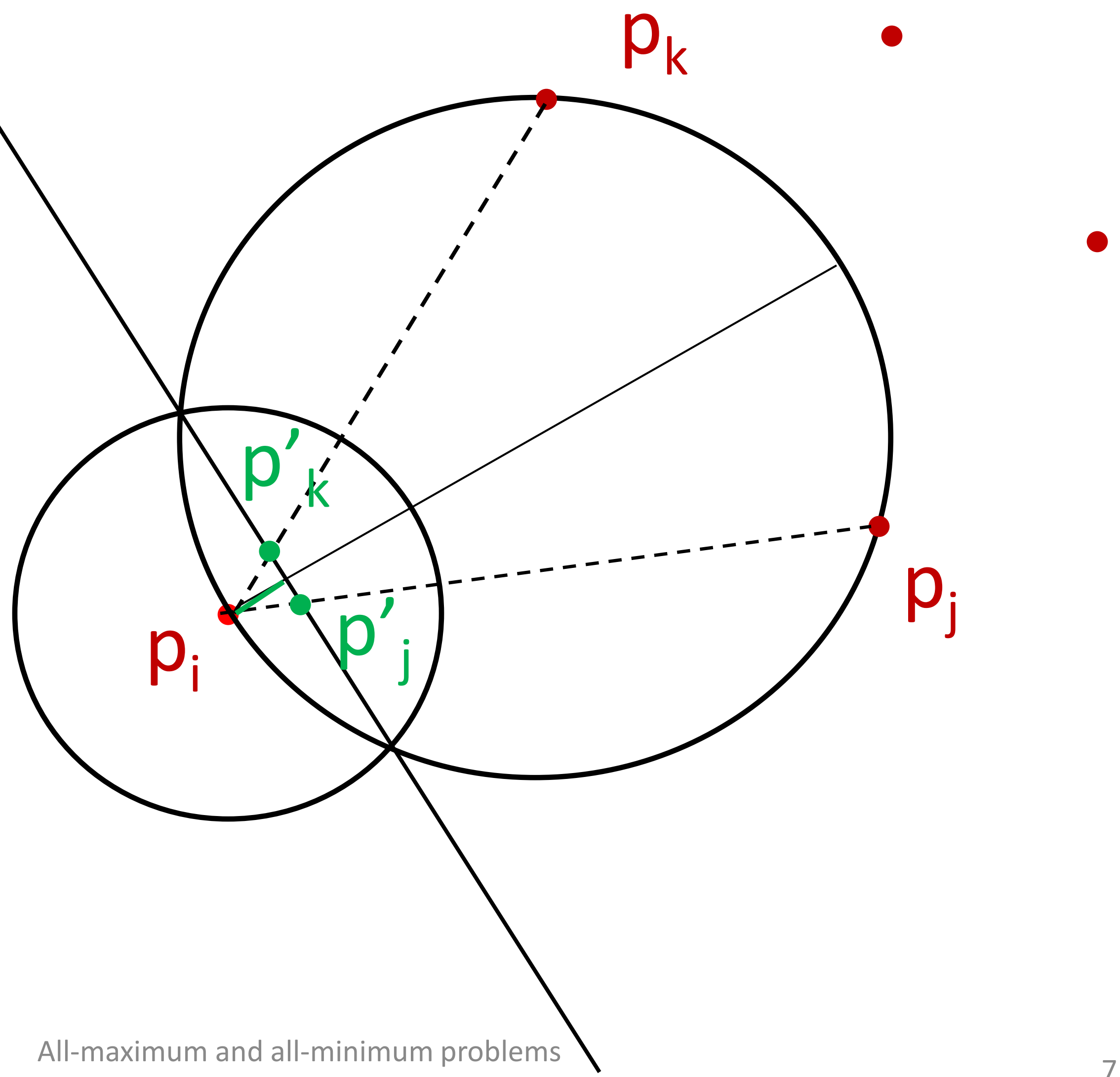
Circumcircle radius Measure: Definition



Circumcircle radius Measure : Definition



Circumcircle radius Measure: Characterization



Maximum Circumcircle radius Measure: Algorithm

- Maximum circumcircle radius problem reduces to finding nearest line from p_i , spanned by a pair points in the inverted set

Minimum Circumcircle radius Measure: Algorithm

- Minimum circumcircle radius problem reduces to finding farthest line from p_i, spanned by a pair points in the inverted set.

Maximum Circumcircle radius Measure: Complexity

- Nearest (farthest) line from p_i : O(n log n) [Daescu et al 2006]
- Nearest (farthest) line from p_i spanned by a pair of points in the inverted set: O(n log n)
- Over all n points: $O(n^2 \log n)$

Conclusions

- Open problems:
  - k-th closest under these different measures (line distance measure already studied by Daescu et al)
  - Better algorithms for the minimum perimeter, circumcircle radius and min-difference measures
  - Optimal algorithms for maximum area and distance measures

References

- A. Aggarwal and J. Park. Notes on searching in multidimensional monotone arrays. In SFCS '88: Proceedings of the 29th Annual Symposium on Foundations of Computer Science, pages 497{512, Wash-ington, DC, USA, 1988. IEEE Computer Society.
- G. Barequet, M. T. Dickerson, and R. L. S. Drysdale. 2-point site voronoi diagrams. Discrete Appl. Math., 122(1-3):37{54, 2002.
- B. Chazelle, L. J. Guibas, and D. T. Lee. The power of geometric duality. BIT, 25:76{90, 1985.
- O. Daescu, J. Luo, and D. M. Mount. Proximity problems on line segments spanned by points. Computational Geometry: Theory and Applications, 33:115{129, 2006.
- R. Drysdale and A. Mukhopadhyay. An o(n log n) algorithm for the all-farthest-segments problem for a planar set of points. Information Processing Letters, 105:47{51, 2008.

References

- K. Duy, C. McAloney, H. Meijer, and D. Rappaport. Closest segments. In Proc. of CCCG 2005, pages 229{231, 2005.
- H. Edelsbrunner, J. O'Rourke, and R. Seidel. Constructing arrangements of lines and hyperplanes with applications. SIAM J. Comput., 15:341{363, 1986.
- H. Edelsbrunner, M. Sharir, and R. Seidel. On the zone therem for hyperplane arrangements. SIAM J. Comput., 22(2):418{429, 1993.
- H. Edelsbrunner and L. Guibas. Topologically sweeping an arrangement. Journal of Computer and System Sciences, 38(1):165{194, 1989.
- A. Gajentaan and M. Overmars. On a class of o(n2) hard problems in computational geometry. Computational Geometry: Theory and Applications, 5(3):165{185, 1995.
- A. Mukhopadhyay, S. Chatterjee, and B. Lafreniere. On the all-farthest-segments problem for a planar set of points. volume 100, pages 120{123, 2006.