Generalized jewels and the point placement problem

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Abstract

The point placement problem is to determine the positions of a linear set of points, $P = \{p_1, p_2, p_3, \ldots, p_n\}$, uniquely, up to translation and reflection, from the fewest possible distance queries between pairs of points. Each distance query corresponds to an edge in a graph, called point placement graph (*ppg*), whose vertex set is P. The uniqueness requirement of the placement translates to line rigidity of the *ppg*.

In this paper we show how to construct in 2 rounds a line rigid *ppg* of size $10n/7 + O(1)$ from small rigid components called 5:5 jewels, which are an extension of the 4:4 jewel of [2]. Though this result is slightly worse than the $4n/3 + O(\sqrt{n})$ upper bound, reported in [1], this is more than offset by the potential for generalization of this construction.

1 Introduction

Recently, many interesting new problems have surfaced at the borderline of computational biology and computational geometry. The point placement problem is one such.

A *ppg* on a linear set of points $P = \{p_1, p_2, p_3, \ldots, p_n\}$ has P as its vertex set, with an edge between p_i and p_j if the distance between p_i and p_j is known. G is said to be line rigid if there is a unique placement of P on a line $\mathcal{L}$, up to translation and reflection, such that the distances between adjacent points is consistent with the edge lengths of G. The point placement problem is to construct a rigid *ppg* from small components that are either line-rigid or remain so under a certain set of sufficient conditions on their edges. In its abstract geometric form, we are given a set of pairwise distances of a set of labeled points on a line and are required to establish their relative linear order uniquely, up to translation and reflection.

In the terminology of learning theory [2] this problem could be restated as learning a set of points on a line, adaptively and non-adaptively. In the latter scheme, learning has to proceed based on a fixed set of given distances, while in the former learning proceeds in rounds, with the edges queried in one round depending on those queried in the previous rounds.

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The computational geometric version of this problem was studied long back by Skiena et al. [5] who gave a practical heuristic for the reconstruction. A polynomial time algorithm was given by Daurat et al. [3].

In computational biology the problem resurfaced in a slightly different form - pairwise distances are known between some pairs of labeled points. The motivation comes from a problem, known as the restriction site mapping. Biologists discovered that certain restriction enzymes cleave a DNA sequence at specific sites known as restriction sites. For example, it was discovered by Smith and Wilcox [6] that the restriction enzyme Hind II cleaves DNA sequences at the restriction sites GTGCAC or GTTAAC. In lab experiments, by means of fluorescent in situ hybridization (FISH experiments) biologists are able to measure the lengths of such cleaved DNA strings. The relative order of the end-points have to be determined as they appear on the line.

Early research on this problem was reported in [4]. In this paper, our first principal reference is [2], where it was shown that the jewel (Fig. 4) and $K_{2,3}$ are both line-rigid, as also how to build large rigid graphs of density 8/5 (this is an asymptotic measure of the number of edges per point as the number of points go to infinity) out of the jewel. Our second principal reference is the work of [1] who improved many of the results of [2], their principal contribution being the 3-round construction of rigid graphs of density 5/4 from 6-cycles and a lower bound on the number of queries necessary in any 2-round algorithm.

In this paper, we determine conditions under which generalizations of the jewel, called $m : n$ jewels, remain line rigid. We also show how to construct in 2 rounds a line rigid point placement graph on n points, using a 5:5 jewel as the basic component. The number of edges queried during this construction is $10n/7 + O(1)$. Though this result is slightly worse than the 4n/3 upper bound, reported in [1], the construction is a lot simpler and transparent, paving the way to possible further improvements.

2 Notations and terminology

Given a *ppg*, G, an assignment of edge lengths is said to be valid if the distances between adjacent points on $\mathcal{L}$ are consistent with these edge lengths. From now on, we will assume this. G, is said to be *line rigid* if there is a unique placement of the vertices on a line for all valid assignments of edge lengths.

Chin et al. [1] introduced the concept of a *layer graph*, which is a kind of graph-drawing. We choose two orthogonal directions x and y (actually, any two non-parallel directions will do) and lay out each edge of the placement graph along one of the two directions, satisfying the following 4 properties:

P1 All edges are parallel to one of the two orthogonal directions, x and y.

P2 The length of an edge is equal to its weight.

P3 Not all edges are along the same direction (thus a layer graph has a two-dimensional extent).

P4 When the layer graph is collapsed onto a line, either to the left or to the right, no two vertices coincide.

Here, we make extensive use of the following useful result they proved.

Theorem 1 *A ppg (G, l) is line rigid iff it cannot be drawn as a layer graph.*

3 Generalized jewels

An $m : n$ jewel consists of a cycle, C_1 of length m and another cycle, C_2, of length n that are joined by a strut going between two vertices Y (of C_1) and Z (of C_2), and hinged at a third common vertex, X (Fig. 1).

In this section, we classify the layer graphs of an $m : n$ jewel, and find conditions that make an $m : n$ jewel line rigid. Finally, we consider concrete examples of the 4 : 4, 5 : 4 and the 5 : 5 jewels.

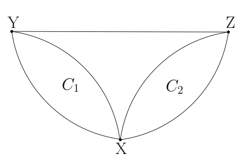

Figure 1: $m : n$-Jewel.

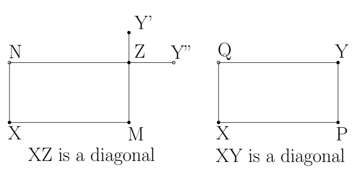

Figure 2: Configuration of L_1 and L_2.

Theorem 2 *Let $L_i = (V_i, E_i)$ denote a layer graph of C_i, when it can be so drawn, or the line on which all its points lie, where $i = 1, 2$. Let $L = (V, E)$ denote a layer graph of an $m : n$ jewel, when it can be so drawn or a line on which all its points lie, where $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2 \bigcup \{YZ\}$.*

1. *If L_1 and L_2 are both lines then so is L.*

2. *If L_1 is a line and L_2 is a layer graph, then L is a layer graph if and only if XZ is not a diagonal of a rectangle in L_2*

3. *If L_2 is a line and L_1 is a layer graph, then L is a layer graph if and only if XY is not a diagonal of a rectangle in L_1*

4. *If both L_1 and L_2 are layer graphs, then L is a layer graph if and only if not both XY and XZ are diagonals of rectangles*

Proof. If L_1 and L_2 are both lines then all the points of L are on a line. Therefore L is not a layer graph.

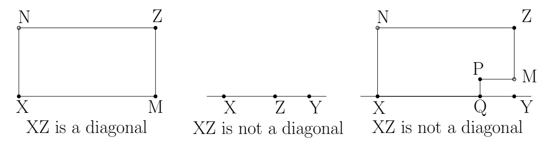

Figure 3: Configurations of L_2.

Consider Case 2. Let XZ be a diagonal of a rectangle (Fig. 3). Since Y and Z are connected, Y must be on line joining M and Z or on the line joining N and Z. But L_1 is a line. Hence X and Y must be along a horizontal line or a vertical line. Therefore Y must coincide with M or N in Fig. 3. Now suppose XZ is not a diagonal of a rectangle. If X and Z are along a horizontal line or a vertical line then we can place all other points of L_1 on this line. If XZ is not a vertical line or a horizontal line (Fig. 3) then we can place Y at the intersection of a vertical line and a horizontal line passing through X and Z such that Y does not coincide with any other point. Then we can place all other points on L_1 on the line passing through XY. By similar arguments, we can prove Case 3.

Finally, assume that both XY and XZ are diagonals of rectangles (Fig. 2) in the layer graphs L_1 and L_2 respectively. If we place Y at Y' then P or Q will coincide with M. If we place Y at Y'' then P or Q will coincide with N. Therefore L is not a layer graph. If both XY and XZ are not diagonals then we can find a layer graph L where no point of L_1 coincides with any other point of L_2.

□

Thus we have the following observation.

Observation 1 *There are three disjoint classes of layer graphs of an $m : n$ jewel (Fig. 1). (1) All the points of C_1 are on a line (C_1 collapses) but C_2 does not collapse and XZ is not a diagonal of a rectangle, (2) All the points of C_2 are on a line (C_2 collapses) but C_1 does not collapse and XY is not a diagonal of a rectangle, (3) Neither C_1 nor C_2 collapses and not both XY and XZ are diagonals of rectangles.*

Theorem 3 *Let C_1 be an m-cycle and let C_2 be an n-cycle (Fig. 1). If there is a layer graph for $m : n$ jewel then either there is a layer graph for C_1 or all the points* *of C_1 are on a line and either there is a layer graph for C_2 or all the points of C_2 are on a line.*

Proof. Let L be a layer graph for an $m : n$ jewel. If we remove from L all the edges corresponding to C_2, all the nodes corresponding to C_2 except X and the edge corresponding to YZ, then the residual graph L_1 must satisfy properties $P1$, $P2$ and $P4$ of a layer graph. If it also satisfies $P3$ then it would be a layer graph of C_1, otherwise all the points of C_1 are on a line. Similarly, we can show that there is a layer graph L_2 for C_2 or all the points of C_2 are on a line L_2. □

Theorem 4 *Suppose we have an $m : n$ jewel. All the conditions that prevent C_1 from being drawn as a layer graph, provided XY is not a diagonal of a rectangle, plus all the conditions that prevent C_2 from being drawn as a layer graph provided XZ is not a diagonal of a rectangle constitute a set of sufficient conditions that make an $m : n$ jewel line rigid.*

Proof. The proof follows easily from Theorem 3 and Observation 1. □

Theorem 5 *The minimum number of conditions for making an n-cycle line rigid is $2^{n-1} - \frac{n^2-n+2}{2}$*

By Theorem 4 and by Theorem 5 we can find $2^{m-1} + 2^{n-1} - \frac{1}{4}\left(3m^2 + 3n^2 - 6m - 6n + 15\right) + \frac{1}{8}[(-1)^{m+1} + (-1)^{n+1}]$ conditions which are sufficient for an $m : n$ jewel to be line rigid.

In this section we show that the 4:4 jewel is line rigid by drawing layer graphs. We then investigate conditions under which $m : n$ jewels remain rigid for different values of n and m.

3.1 4-cycle to 4 : 4 Jewel

Observation 2 *A four cycle $XAYB$ is line rigid if $|XA| \neq |YB|$*

A proof of the above fundamental observation can be found in [2]. Figure 4 shown below is called a jewel in [2]. In view of our subsequent generalizations, we call it instead 4 : 4 jewel.

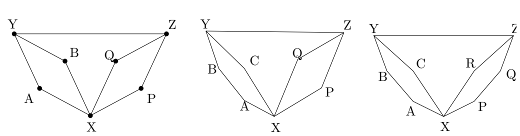

Figure 4: Point placement graphs for 4 : 4, 5 : 4 and 5 : 5 jewels.

Theorem 6 *The 4 : 4 jewel is line rigid.*

Proof. We can easily verify that there is no layer graph that satisfies any of the condition of Observation 1. Therefore there is no layer graph for the 4 : 4 jewel. □

3.2 5-cycle to 5 : 4-Jewel to 5 : 5-Jewel

We will use theorem 4 to find the conditions that makes a 5 : 4 jewel line rigid. There are five classes of layer graphs of a 5 cycle $XABYC$. We get one condition from each class of layer graph (Fig. 5). For two of those classes we see that XY is a diagonal of a rectangle. Hence we will get three conditions that will prevent a 5 cycle from being a layer graph where XY is not a diagonal. There is no layer graph of the 4 cycle $XPZQ$ where XZ is not a diagonal. Therefore, by the Theorem 4 we can say that $|XC| \neq |AB|$, $|XA| \neq |YB|$ and $|YC| \neq |AB|$ are sufficient for the 5:4 jewel in figure 4 to be line rigid.

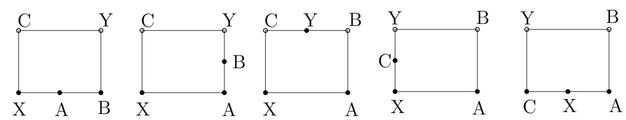

Figure 5: Classes of layer graphs of 5-cycle.

Similarly we can find the conditions for a 5:5 jewel by using Theorem 4. We will get three conditions from each 5 cycle. Therefore there are six conditions that will make a 5:5 jewel line rigid. These conditions are $|XC| \neq |AB|$, $|XA| \neq |YB|$, $|YC| \neq |AB|$, $|XR| \neq |PQ|$, $|ZR| \neq |PQ|$ and $|XP| \neq |ZQ|$.

4 A 2-round algorithm for a 5:5 jewel

This algorithm makes crucial use of the following observation :

Observation 3 *At a point p on a line there can be at most two edges incident that have the same length.*

We want to make the 5:5 jewel line rigid irrespective of the values of distances $|AB|$, $|XC|$, $|PQ|$ and $|XR|$ so that we can query the distances in such a way that the rigidity conditions are satisfied. For that we need to reformulate the four conditions $|XC| \neq |AB|$, $|YC| \neq |AB|$, $|XR| \neq |PQ|$ and $|ZR| \neq |PQ|$ with other suitable equivalent ones.

From Figure 5 we see that we can replace the condition $|XC| \neq |AB|$ by the equivalent condition $|XA| \neq |YB| \pm |YC|$. Similarly, we can replace $|XR| \neq |PQ|$ by $|XP| \neq |ZQ| \pm |ZR|$.

To reformulate the condition $|YC| \neq |AB|$ we draw the layer graphs of the whole jewel in such a way that it violates this condition (Figs. 6(a) to 6(d)). From these figures we see that we can replace this condition by the conditions: $|XP| \neq |ZQ| \pm |YC| \pm |YZ|$ and $|XP| \neq |ZQ| \pm |ZR|$ (Fig. 6(a)), $|ZR| \neq |YC|$ (Fig. 6(b)), $|XA| \neq |ZR| \pm |YB| \pm |YZ|$ (Fig. 6(c)), and $|XA| \neq |YB| \pm |YZ|$ (Fig. 6(d)). Among them $|XP| \neq |ZQ| \pm |ZR|$ has already been found. We have four new conditions such as $|XP| \neq |ZQ| \pm |YC| \pm |YZ|$, $|ZR| \neq |YC|$, $|XA| \neq |ZR| \pm |YB| \pm |YZ|$ and $|XA| \neq |YB| \pm |YZ|$. Similarly, to replace the condition $|ZR| \neq |PQ|$ we need the four conditions: $|XA| \neq |ZR| \pm |YB| \pm |YZ|$, $|ZR| \neq |YC|$, $|XP| \neq |ZQ| \pm |YC| \pm |YZ|$ and $|XP| \neq |ZQ| \pm |YZ|$. Among these two groups $|XA| \neq |ZR| \pm |YB| \pm |YZ|$, $|ZR| \neq |YC|$ and $|XP| \neq |ZQ| \pm |YC| \pm |YZ|$ are common.

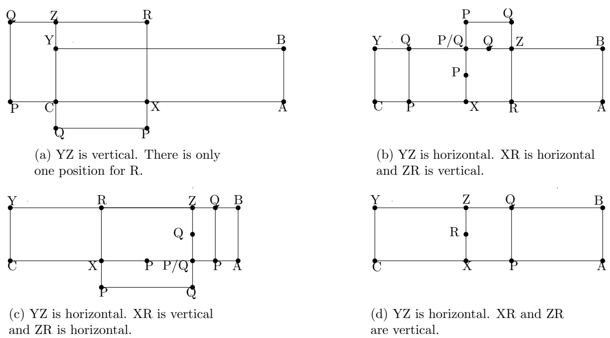
(a) YZ is vertical. There is only one position for R.
(b) YZ is horizontal. XR is horizontal and ZR is vertical.
(c) YZ is horizontal. XR is vertical and ZR is horizontal.
(d) YZ is horizontal. XR and ZR are vertical.

Figure 6: Replacing condition $|YC| \neq |AB|$.

Thus, we have the following nine distinct conditions to make the 5:5 jewel line rigid: $|YB| \neq |XA|$, $|YB| \neq |XA| \pm |YZ|$, $|YC| \neq |YB| \pm |XA|$, $|ZQ| \neq |XP|$, $|ZQ| \neq |XP| \pm |YZ|$, $|ZQ| \neq |YC| \pm |XP| \pm |YZ|$, $|ZR| \neq |YC|$, $|ZR| \neq |ZQ| \pm |XP|$ and $|ZR| \neq |YB| \pm |XA| \pm |YZ|$.

Algorithm 1. For convenience we change the labels as follows: $X \rightarrow X_i$, $A \rightarrow A_i$, $B \rightarrow B_j$, $C \rightarrow B_k$, $P \rightarrow P_i$, $Q \rightarrow Q_m$ and $R \rightarrow Q_l$. Let the total number of points be $n = 7b + 40$, where b is a positive integer. In the first round, we choose $6b + 39$ distance queries represented by the edges in the graph in Fig. 7. There are $2b + 10$ children $B_j (j = 1, ..., 2b + 10)$ rooted at Y and $2b + 28$ children $Q_l (l = 1, ..., 2b + 28)$ rooted at Z. We group the remaining $3b$ nodes into groups of 3 as (A_i, X_i, P_i) $(i = 1, ..., b)$ and query the distances $|A_iX_i|$ and $|X_iP_i|$, $(i = 1, ..., b)$.

In the second round, for each 2-link (A_iX_i, X_iP_i) we find edges YB_j, YB_k rooted at Y and ZQ_l, ZQ_m rooted at Z to form an extended jewel which satisfy the conditions for rigidity. Then we query the distances $|A_iB_j|$, $|X_iB_k|$, $|X_iQ_l|$ and $|P_iQ_m|$. The 10 additional children at Y and the 28 additional children at Z provide us with the latitude to choose edges that satisfy the above conditions for rigidity. Using Observation 3, we can show that a maximum of 38 edges do not satisfy the conditions for rigidity. So, for each 2-link (A_iX_i, X_iP_i) we can always find edges YB_j, YB_k, ZQ_l and ZQ_m for the extended jewel such that the conditions for rigidity are satisfied.

For each of the unused 10 leaves B_j of node Y we query the distance B_jZ. Similarly, for each of the 28

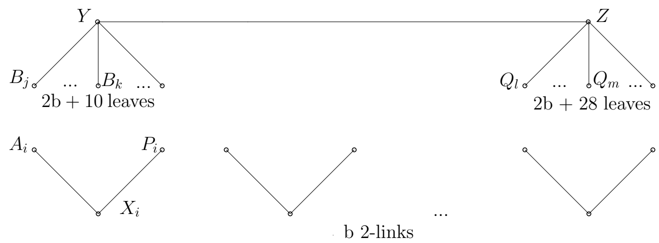

Figure 7: Queries in the first round for 5:5 jewel.

unused leaves of Z we query its distance from Y.

We need $6b+39$ queries in the first round and $4b+38$ queries in the second round. In total $10b + 77$ pairwise distances are to be queried for the placement of $7b + 40$ points. Thus, $10n/7+O(1)$ queries are sufficient to place n distinct points on a line using two rounds.

5 Conclusions

It would be interesting to know if extending the results to 6:6, 7:7 and 8:8 jewels improve the upper bounds for two or more rounds. Preliminary indications seem positive. Another interesting direction is to consider learning a set of points in the plane. We have not seen any published work on this topic.

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