A note on the approximation ratio for computing an approximate spanning ellipse in the streaming model

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Abstract

An algorithm for computing an approximate elllipse in the streaming model was proposed by Mukhopadhyay and Greene [1]. Based on experimental results it was conjectured in that paper that the ratio of the areas of the approximate to the exact ellipse is bounded, lying between 5 and 6. In this note, we construct an example to show that the approximation ratio could become unbounded.

1 Introduction

In [2], Zarrabi-Zadeh and Chan proposed a simple algorithm for computing an approximate spanning ball of a set of n points $P = \{p_1, p_2, p_3, \cdots, p_n\}$ in d-dimensions in the streaming model of computation. The approximation ratio of the volume of the approximate spanning ball to the exact one for this algorithm was shown to be $9/4$, which is tight. Inspired by this result, an algorithm in the same streaming model was proposed by Mukhopadhyay and Greene [1] for computing an approximate minimum spanning ellipse (see `http://cs.uwindsor.ca/~asishm` for a beautiful implementation of this algorithm by Eugene Greene). Based on extensive experimental evidence, they conjectured that the ratio of the area of the approximate ellipse to the area of the exact ellipse lies between 5 and 6.

In this note we have constructed an example input sequence to show that the approximation ratio could become unbounded.

2 Streaming algorithm for an approximate ellipse

To make the paper self-contained, we include a brief discussion of the approximation algorithm of Mukhopadhyay and Greene [1]. Their algorithm goes as follows.

Given the current approximate ellipse E_i and a point p_i outside it, they construct an elliptic transformation that carries E_i into a unit circle; the same transformation is applied to p_i, with an additional rotation that carries the transformed point onto the x-axis in the tranformed plane. Next, using elementary algebraic tehniques, it is shown how to construct a minimum ellipse that contains the unit circle and goes through the transformed point. An inverse elliptic transformation gives us the minimum approximate ellipse E_{i+1} in the original plane. The utility of the elliptic transforamtion is that it preserves relative areas as well as their ratios.

In the following section we construct an example input sequence to show that the approximation ratio can be unbounded.

3 An example with unbounded approximation ratio

To construct the example we need the following important lemma.

Lemma 1 *The exact ellipse incident on* $(0, k)$, $(0, -k)$ *and* $(3l, 0)$ *must also pass through* $(-l, 0)$. *Moreover, the length of the minor axis of this exact ellipse is* $\frac{4k}{\sqrt{3}}$.

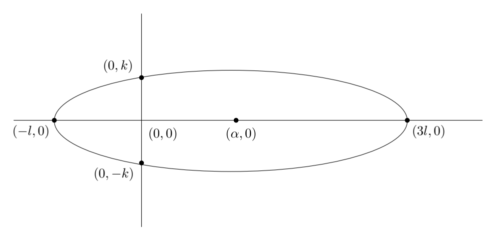

Figure 1: Exact ellipse through $(0, k)$, $(0, -k)$ and $(3l, 0)$

Proof. Let $\frac{(x-\alpha)^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of the exact ellipse. Since it passes through $(3l, 0)$, we have $a = 3l - \alpha$. By substituting $(0, k)$ we get $b = \frac{ak}{\sqrt{a^2 - \alpha^2}}$. The area of this ellipse $A = \pi ab = \pi \frac{(3l-\alpha)^2 k}{\sqrt{(3l-\alpha)^2 - \alpha^2}}$. By setting $\frac{dA}{d\alpha} = 0$, we get $\alpha = l, \frac{3l}{2}, 3l$. But $\alpha = l$ gives the minimum area. The center of the exact ellipse is $(l, 0)$, and $a = 3l - \alpha = 2l$. Therefore it passes through $(-l, 0)$. Since $\alpha = l$ and $a = 2l$, we get $b = \frac{ak}{\sqrt{a^2 - \alpha^2}} = \frac{2lk}{\sqrt{4l^2 - l^2}} = \frac{2k}{\sqrt{3}}$. Thus the length of the minor axis is $\frac{4k}{\sqrt{3}}$. □

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In the following lemma we construct an example for which the ratio of the length of the minor axis of approximate ellipse to the length of the minor axis of exact ellipse can be made unbounded. We make crucial use of Lemma 1.

Lemma 2 *There exists an input point sequence for which the ratio of the minor axis of approximate ellipse to that of the exact ellipse becomes unbounded.*

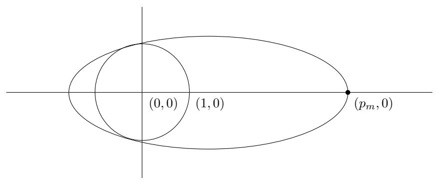

Figure 2: m-th transformed plane P_m

Proof. Consider three points $(0,-1)$, $(0,1)$ and $(-1,0)$. Let E_0 be the exact ellipse through these three points. Since the approximate ellipse through these 3 points is identical with the exact one, the ratio of their areas is 1 in this case. Now we start adding points on the positive x-axis.

If we add a point $(p_m, 0)$ on the positive x-axis then the exact ellipse will pass through $(0,-1)$, $(0,1)$, and $(p_m, 0)$. For any p_m, the length of the minor axis of the exact ellipse is $\frac{4}{\sqrt{3}}$ (Lemma 1).

We denote the original plane by P_0. The transformation S_0T_0 will transform E_0 to the unit circle where T_0 is a translation and S_0 is a scaling. Let $P_1 = S_0T_0P_0$ be the transformed plane. We can choose a point $(p', 0)$ on this transformed plane such that the minor axis of the spanning ellipse, which passes through $(p', 0)$ and encloses the unit circle, is greater than 1. Suppose ST is the transformation which transforms this new ellipse to the unit circle where S is a scaling and T is a translation. Let $P_2 = STP_1$. Take $(p', 0)$ on P_2 and find the ellipse that passes through $(p', 0)$ and encloses the unit circle.

If we repeat this process n times, then each time we increase the minor axis of the ellipse same amount. Denote the approximate ellipse on P_n by E_n. The approximate ellipse on the original plane can be found by applying the transforamtion $(T^{-1}S^{-1})^n T_0^{-1} S_0^{-1} E_n$. Since S is a shrinking along y-axis, S^{-1} is an expansion along y-axis. Therefore we can expand the minor axis of approximate ellipse as much as we want by adding a sufficient number of points. Since the minor axis of the exact ellipse is fixed, we conclude that the ratio of the minor axis of approximate ellipse to that of the exact ellipse is unbounded.

Note that the point $(p', 0)$ on the m-th transformed plane corresponds to $(T^{-1}S^{-1})^m T_0^{-1} S_0^{-1}(p', 0)$ on the original plane. $\square$

Lemma 3 *The ratio of the length of major axis of approximate ellipse to the length of major axis of exact ellipse, constructed by the method described in Lemma 2, is greater than* $3/4$.

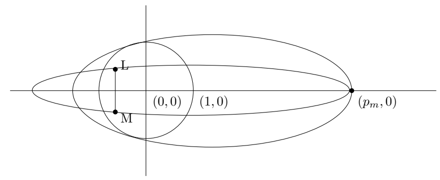

Figure 3: LM is the transformed line segment joining $(0,-1)$ and $(0,1)$ on the original plane

Proof. Suppose $2a_A$ and $2a_E$ denote the length of the major axis of the approximate ellipse and the exact ellipse respectively. Figure 3 shows the m-th transformed plane P_m, and the line segment LM is the transformed line segment joining $(0,1)$ and $(0,-1)$ on the original plane. By Lemma 1 the length of the major axis is $4/3$ of the distance of $(p_m, 0)$ from LM. Since LM must be enclosed by the unit circle, the maximum distance of point $(p_m, 0)$ from LM is $p_m + 1$. Hence $2a_E \leq \frac{4}{3}(p_m + 1)$. From Figure 3 it is clear that $2a_A \geq p_m + 1$. This implies that $\frac{2a_A}{2a_E} \geq \frac{p_m + 1}{(4/3)(p_m+1)} \geq \frac{3}{4}$. $\square$

Theorem 4 *There exists an input point sequence of points for which the approximation ratio of the ellipse areas becomes unbounded.*

Proof. Let a_A, b_A, and Area_A denote the semi-major axis, semi-minor axis and the area of the approximate ellipse respectively in the transformed plane, while a_E, b_E, and Area_E denote the same quantities respectively for the exact ellipse. By the method described in Lemma 2 we can construct an input point sequence for which $\frac{b_A}{b_E}$ is unbounded. By Lemma 3 we have $\frac{a_A}{a_E} \geq \frac{3}{4}$. Since relative areas are preserved by an elliptic transformation, the ratio $\frac{\text{Area}_A}{\text{Area}_E}$ is identical for the ellipses in the original plane. Now $\frac{\text{Area}_A}{\text{Area}_E} = \frac{\pi a_A b_A}{\pi a_E b_E} = \frac{a_A}{a_E}\frac{b_A}{b_E} \geq \frac{3}{4}\frac{b_A}{b_E}$. Since by Lemma 2, there exists an input sequence for which the ratio $\frac{b_A}{b_E}$ is unbounded, we conclude that $\frac{\text{Area}_A}{\text{Area}_E}$ is unbounded for this input sequence. $\square$

Consider a sequence of ellipses described in Lemma 2. Suppose for E_0, the lengths of the axis parallel to x-axis and y-axis are r_x and r_y respectively. The scaling factors to transform E_0 to a unit circle are $1/r_x$ and $1/r_y$ along x-axis and along y-axis respectively. Suppose on the transformed planes $P_1, P_2, P_3, \cdots$, the minimum ellipses $E_1, E_2, E_3, \cdots$ that are enclosing the unit circle and the point $(p_m, 0)$ has the minor axis $1+s$ and the major axis p_m+1+t. On the original plane the minor axis and the major axis of E_n are $r_x(p_m+1+t)^n$ and $r_y(1+s)^n$ respectively. In the following theorem we will show that the approximation ratio does not depend on the eccentricity.

Theorem 5 *The approximation ratio does not depend on the eccentricity of the exact ellipse.*

Proof. Suppose the eccentricity of an exact ellipse e and a positive real number M are given. We will construct an example such that the approximation ratio is greater than or equal to M.

Choose an n such that $\frac{(1+s)^{n-1}}{p_m+1+t} > 2M$ and choose an a such that $(p_m+1+t)^{n-1} < a < (p_m+1+t)^n$. Then $b = a\sqrt{1-e^2}$. Let $k = \frac{\sqrt{3}b}{2}$.

Start with three points $(0,-k)$, $(0,k)$ and $(-1,0)$ and construct a sequence of ellipses described in Lemma 2. The lengths of the major axis and the minor axis of E_{n-1} on the original plane are $a_A = r'_x(p_m+1+t)^{n-1}$, $b_A = r'_y(1+s)^{n-1}$ respectively. The length of the major axis and the minor axis of the exact ellipse are $a_E = a$ and $b_E = \frac{2k}{\sqrt{3}}$. Therefore,

$$
\begin{aligned}
\frac{\text{Area}_A}{\text{Area}_E} &= \frac{\pi a_A b_A}{\pi a_E b_E} \\
&= \frac{a_A}{a_E}\frac{b_A}{b_E} \\
&= \frac{r'_x(p_m+1+t)^{n-1}}{a}\frac{r'_y(1+s)^{n-1}}{\frac{2k}{\sqrt{3}}} \\
&= \frac{r'_x(p_m+1+t)^n}{a}\frac{\sqrt{3}r'_y(1+s)^{n-1}}{2k(p_m+1+t)} \\
&= r'_x\frac{(p_m+1+t)^n}{a}\cdot\frac{\sqrt{3}}{2}\cdot\frac{r'_y}{k}\cdot\frac{(1+s)^{n-1}}{(p_m+1+t)} \\
&\geq \frac{1}{2}\cdot 1\cdot 1\cdot 1\cdot 2M \\
&= M
\end{aligned}
$$

□

In the next theorem we will show that for any given exact ellipse we can find a sequence of points for which the approximation ratio could be as large as we want.

Theorem 6 *For any given exact ellipse there is a sequence of points for which the approximation ratio is arbitrarily large.*

Proof. Suppose an exact ellipse is given on a plane P_{-2}. Let $M \in \mathbb{R}$. Using appropriate rotation and translation align the major axis and the minor axis of this exact ellipse along x-axis and y-axis respectively. Let denote this plane by P_{-1} and let e be the eccentricity of the exact ellipse.

Now determine a and b which are described in the proof of Theorem 5. Using the appropriate scaling factor transform the exact ellipse to the ellipse with the major axis and the minor axis being a and b respectively. Denote this plane by P_0. Determine a sequence of points described in the proof of Theorem 5. By transforming this sequence of points to P_{-2} plane we get the desired sequence. □

We do not need a sequence of points to show that the approximation ratio is unbounded. We can show it by only 4 points. From [1] we know that the minimum ellipse that encloses the unit ball and the point $(d,0)$ has the equation $a(x-x_0)^2 + cy^2 = 1$ where

$$
\begin{aligned}
x_0 &= \frac{d^2-1}{3d-\beta-\beta^{-1}} \\
a &= \frac{1}{(\beta-x_0)(\beta-1-x_0)} \\
c &= \frac{1}{1-x_0\beta} \\
\beta &= \frac{d-\sqrt{d^2+8}}{4}
\end{aligned}
$$

The area of this minimum ellipse is

$$A_{\text{appr}} = \pi\frac{(d-\beta)(d\beta-1)^2}{(\beta^2-2d\beta+1)^{3/2}}$$

.

The approximation ratio is 1 for the ellipse through $(0,1)$, $(0,-1)$, and $(1,0)$. After transforming this ellipse to the unit circle we will add a new point on the x-axis of the transformed plane. Suppose the coordinates of this new point is $(d,0)$ on the transformed plane. The minor axis of the exact ellipse on the transformed plane is bounded and the length of the major axis is at most $\frac{2}{3}(d+1)$ by Lemma 1. Therefore,

$$\frac{A_{\text{appr}}}{A_{\text{exact}}} = \frac{(d-\beta)(d\beta-1)^2}{k(d+1)(\beta^2-2d\beta+1)^{3/2}}$$

This is a function of d. The numerator of this expression has degree 5 and the denominator has degree 4. Therefore, we can make this ratio as large as we want by taking the point far away.

4 Conclusions

Since the ratio of areas is not bounded, the important unresolved question is: What is the ratio that remains bounded ? We conjecture that there always exists some direction in which the ratio of the widths of the ellipses remains bounded. Thus for the example above, the ratio of the major axes is bounded above by 3. That this is plausible is supported by the result of [2] for approximate balls. We have made some progress towards resolving this question.

References

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[2] H. Zarrabi-Zadeh and T. Chan. A simple streaming algorithm for minimum enclosing balls. In *Proceedings of the 18th Canadian Conference on Computational Geometry (CCCG'06)*, pages 139–142, 2006.

5 Acknowledgements

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